MATH 695 11/16/2022

Poincaré duality: If M is a connected compact smooth n-manifold orientable with respect to $H^*(?;R)$

commutative ring

condition vacuous for $R = \mathbb{Z}/2$

geom. orientability implies it for any R

equivalent for $\mathbb{Z}$

R-orientable

$$H_k(M;R) \cong H^{n-k}(M;R).$$

We can say a bit more about the isomorphism.

Assume $H_*(M;R)$ is a free R-module. Then it is finitely generated as an R-module. $H_n(M;R) \cong H^0(M;R) = R$

$$\cup : H^k(M;R) \otimes H^{n-k}(M;R) \longrightarrow H^n(M;R) = R$$

by the UCT, these are also free finitely generated R-modules.

bilinear pairing

Another version of Poincaré duality says that it is a perfect pairing.

For f.g. free R-modules M, N (R commutative ring)

$$\mu : M \otimes N \to R$$

is called a perfect pairing when its adjoint

$$M \xrightarrow{\sim} \mathrm{Hom}_R(N, R) =: N^\vee$$

is an isomorphism.

we actually have strong duality in f.g. R-modules, so this is DN.

HW ③ Prove that for a bilinear pairing $\langle\,,\rangle : M \otimes N \to R$ of f.g. free R-modules, the adjoint

$$M \to \mathrm{Hom}_R(N, R)$$

is an isomorphism if and only if the adjoint

$$N \to \mathrm{Hom}_R(M, R)$$

is an isomorphism

Back to Poincaré duality. Assume M compact connected R-orientable smooth n-manifold, $H_k(M;R)$ (hence $H^k(M;R)$) is a free R-module.

depends on orientation

$$\cup : H^k(M;R) \otimes_R H^{n-k}(M;R) \to H^n(M;R) \cong R$$

is a perfect pairing.

__Corollary:__ $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$ $\qquad \deg(x) = 2$

$H^*(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[y]/(y^{n+1})$ $\qquad \deg(y) = 1$

$\therefore$ $H^*(\mathbb{C}P^\infty;\mathbb{Z}) = \mathbb{Z}[x]$

$H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) = \mathbb{Z}/2[y]$.

__Proof by induction:__

$\mathbb{C}P^n \subset \mathbb{C}P^{n+1}$

$(2n+1)$-equivalence

homomorphism of rings

$H^*(\mathbb{C}P^{n+1};\mathbb{Z}) \to H^*(\mathbb{C}P^n;\mathbb{Z})$

Additively

$$
\begin{array}{rcccl}
 & 1 & \longmapsto & 1 & \text{suppose we know} \\
\text{degree } 0 \to & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \cong \mathbb{Z}[x]/x^n \\
 & 0 & & 0 & \text{contains } 1 \\
 & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \\
 & x & \longmapsto & x & \\
 & \vdots & & \vdots & \\
\text{degree } n \to & \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} & \\
 & x^n & \longmapsto & x^n & \\
 & 0 & & 0 & \\
 & \mathbb{Z} & & 0 &
\end{array}
$$

$\cup : H^{2n}(\mathbb{C}P^{n+1};\mathbb{Z}) \otimes H^2(\mathbb{C}P^{n+1};\mathbb{Z}) \to \mathbb{Z}$

call this x

is a perfect pairing

$x^{n+1} = x^n \cup x$ generates $H^{2n+2}(\mathbb{C}P^{n+1};\mathbb{Z})$ $\}$ $\therefore$ x^{n+1}

Example: $\mathbb{RP}^{2n+1}$ is an orientable manifold

We have Poincaré duality with coefficients in $\mathbb{Z}$:

degree	$H_k(\mathbb{RP}^{2n+1};\mathbb{Z})$	$H^k(\mathbb{RP}^{2n+1};\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
⋮	$\mathbb{Z}/2$	0
⋮	0	$\mathbb{Z}/2$
⋮	⋮	0
⋮	⋮	⋮
	$\mathbb{Z}/2$	⋮
$2n$	0	$\mathbb{Z}/2$
$2n+1$	$\mathbb{Z}$	$\mathbb{Z}$

It makes <u>no sense</u> to say that we have a perfect pairing $H^k \otimes H^{n-k} \to \mathbb{Z}$.

(mistake in Poincaré's proof).

Spectra (needed to discuss duality in generalised homology and cohomology).

Desiderata for the derived category DSpectra:

- Σ (suspension) should be an equivalence of categories
- There should be products and coproducts
- universal with these requirements.

It is a disadvantage to work in the derived category only because derived categories do not have equalisers or coequalisers (except in trivial cases which do not apply here), so we couldn't even construct the mapping cone.

We want more structure than just the derived category.

The fashionable term for the extra information we need: <u>∞-category</u>

The traditional term: "<u>a point set category</u>"