MATH 695 11/18/2022

What is an ∞-category?

<u>Classically</u>: A category $\mathcal{C}$ where on $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ for $X, Y \in \mathrm{Obj}\,\mathcal{C}$, there is a topology and composition is continuous.

Point-set topology is quirky (compact generatedness, ...)

There have always been efforts to make the treatment of topology <u>combinatorial</u>. (Caveat: G-equivariant where G is a compact Lie group)

What is the combinatorial approach? (Power up the notion of a simplicial complex.)

Study the functoriality of the standard simplices $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1,\ x_i \geq 0\}$

Face maps $\partial_i : \Delta^n \to \Delta^{n+1}$ $\quad 0 \leq i \leq n$

$$(x_0, \dots, x_n) \longmapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_n)$$

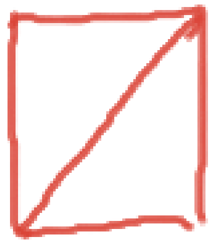
Degeneracy maps

$$s_i : \Delta^n \to \Delta^{n-1}, \quad 1 \leq i \leq n$$

$$(x_0, \dots, x_n) \longmapsto (x_0, \dots, x_{i-2}, x_{i-1} + x_i, x_{i+1}, \dots, x_n)$$

improves the behaviour under products

The compositions of these maps is equivalent to the category



Δ where obj $\Delta = \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\mathrm{Mor}_\Delta(m, n)$ = non-decreasing maps $\{0, \dots, m\} \to \{0, \dots, n\}$

A simplicial set is a functor $\Delta^{op} \to Sets$.

A simplicial object in any category $\mathcal{C}$: a functor $\Delta^{op} \to \mathcal{C}$.

$\underbrace{\text{simplicial objects make a category}}_{\Delta^{op}-\mathcal{C}.}$ (morphisms = natural transformations).

The idea of a simplicial set generalising a simplicial complex:

Geometric realisation of a simplicial set $S_\cdot : \Delta^{op} \to Sets$:

$$|S_\cdot| = S_\cdot \times_{\Delta} \Delta^{\cdot}$$

$$\Delta^{\cdot} : \Delta \to Top$$
$$n \mapsto \Delta^n$$

$$= \mathrm{coeq}\left(S_\cdot \times_{\mathrm{Obj}\,\Delta} \mathrm{Mor}\,\Delta \times_{\mathrm{Obj}\,\Delta} \Delta^{\cdot} \rightrightarrows S_\cdot \times_{\mathrm{Obj}\,\Delta} \Delta^{\cdot} \right)$$

<u>Theorem</u> (Milnor): $\underbrace{|S_\cdot \times T_\cdot|}_{(S_\cdot \times T_\cdot)_n = S_n \times T_n} = |S_\cdot| \times |T_\cdot|$. $\square$

(Needs degeneracies)

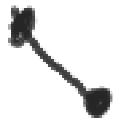
We can build D Top just from the category Δ^{op}-Set (Quillen).

$$f : S_\cdot \to T_\cdot$$

is called an <u>equivalence</u> if $|f| : |S_\cdot| \to |T_\cdot|$ is a weak equivalence.

(there is an alternative definition not using topology)

Homotopy: $f, g : S_\cdot \to T_\cdot$ $\qquad h : \Delta^1 \times S_\cdot \longrightarrow T_\cdot$

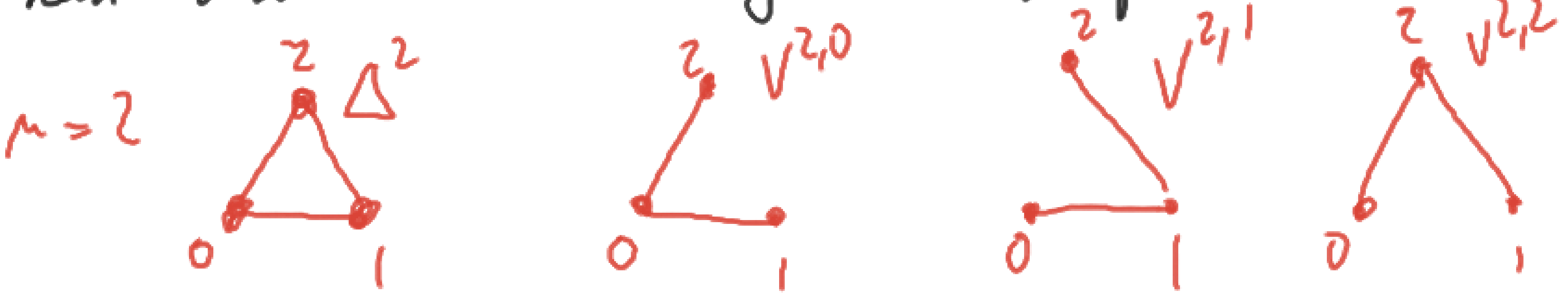


$$\Delta^k_n = \mathrm{Hom}_\Delta(n,k)$$

Not an equivalence relation, the relation of homotopy $\simeq$ is the smallest equivalence relation containing the existence of homotopy between f, g.

The category of simplicial sets, homotopy classes of morphisms: $h\Delta^{op}$-Sets.

The category $h\Delta^{op}$-Set has localization with respect to Kan complexes which are simplicial sets S. which satisfy the Kan condition: We can form a simplicial set $V^{n,k} = \Delta^n \setminus$ the top simplex and its kth face.



The Kan condition means that any morphism $V_\cdot^{n,k} \to S_\cdot$ extends to $\Delta_\cdot^n$.

$$
\begin{tikzcd}
V_\cdot^{n,k} \arrow[r, "\text{given}"] \arrow[d, hook] & S_\cdot \\
\Delta_\cdot^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}
$$

<u>Theorem</u>: Simplicial sets have localisation with respect to Kan complexes, the derived category is equivalent to DTop.

<u>Observation</u>: A minimal Kan complex is a simplicial set satisfying the Kan condition <u>with uniqueness</u>. <u>Theorem</u>: $\{\cong$ classes of minimal Kan complexes$\}$ in Δ^{op}-sets

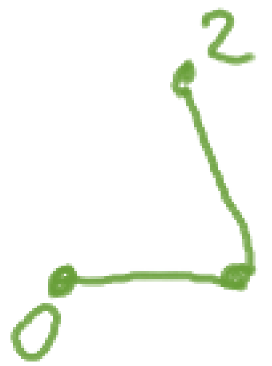
$\cong \{\cong$ classes in DTop$\}$ weak homotopy types

Finally, back to our subject: A *quasi-category* is a simplicial set that satisfies the Kan condition partially:

$$\begin{tikzcd} V^{n,k}_{\cdot} \arrow[r, "\text{given}"] \arrow[d, hook] & S_{\cdot} \\ \Delta^n_{\cdot} \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

only for $1 \leq k < n$

$n = 2$, only $k = 1$



(the "1-morphisms" in a quasicategory are the 1-simplices). Partial Kan condition represents (non-unique) composition

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In some sense, the notion of an ∞-category (as presented in the beginning) is "equivalent" to the notion of a quasicategory.

passages both ways, no general framework to make a formal statement.

Most people who use "∞-category" mean quasicategory.

J. Lurie: Derived Topos Theory, Higher Algebra

NO HW today

Next time: Spectra