MATH 695 11/21/2022

The ∞-category of spectra

category with topology on morphism sets - topological category

May spectra: $Z = (Z_n)_{n \in \mathbb{N}_0}$ of based spaces ← compactly generated weakly Hausdorff

$$\rho_n : Z_n \xrightarrow{\simeq} \Omega Z_{n+1}.$$

Morphisms: $f: Z \to Z'$ are tuples of based continuous maps $f_n : Z_n \to Z'_n$ } Sp

$$\begin{array}{ccc} Z_n & \xrightarrow{f_n} & Z'_n \\ \rho_n \downarrow & & \downarrow \rho_n \\ \Omega Z_{n+1} & \xrightarrow{\Omega f_{n+1}} & Z'_{n+1} \end{array}.$$

Pre-Sp

Pre-spectra: same definition (objects and morphisms) except omit the condition that ρ_n be homeomorphisms.

Forgetful functor $\mathbb{I}\text{-}Sp \to$ Pre-Sp has a left adjoint L: Pre-Sp $\to$ Sp (spectrification)

a general theorem of Freyd and Kelly

LNM 1213 (Lewis-May-Steinberger) compact Lie group equivariant

Remark: Sp is equivalent to the category of $(Z_n)_{n\in Q}$ where $Q \subseteq \mathbb{Z}$ is any cofinal set with respect to $\subseteq$ (contains arbitrary large elements)

If n' is the successor of n in Q

$\rho_{n,n'} : Z_n \to \Omega^{n'-n} Z_{n'}$

(fill in intermediate $Z_{n''}$ by taking $n<n''<n'$ by taking $\Omega^{n'-n''} Z_{n'}$)

For Pre-Sp, these categories are not equivalent, we have categories Pre-Sp_Q, spectifications L_Q on them.

Pre-Sp_Q have limits and colimits level-wise,

Sp also have limits and colimits. Limits are done level-wise, colimits are done level-wise and then applying spectification L.

this leads only to a pre-spectrum

<u>Example</u>: There is a functor $\Omega^\infty : \mathrm{Sp} \to \mathrm{Based}$. This has a left adjoint

$$(Z_n) \mapsto Z_0$$

$\Sigma^\infty : \mathrm{Based} \to \mathrm{Sp}$: $X \longmapsto$ prespectrum $D = (D_n)$ $D_n = \Sigma^n X$ $\Sigma(\Sigma^n X) \overset{=}{\to} \Sigma^{n+1} X$ (Ω adjoint to)

$$\Sigma^\infty X := LD.$$

Note: Spectra have a <u>shift functor</u> $[k]$

$$Z = (Z_n)$$

$$Z[k]_n = (Z_{n-k})$$

This is an equivalence of categories.

	$\vdots$	$\vdots$
		$\mathbb{C}P^\infty$
	$\mathbb{C}P^\infty$	S^1
	S^1	$\mathbb{Z}$
$\to 0$	$\mathbb{Z}$	0
	0	0
	0	$\vdots$
	0	
	$\vdots$	$\vdots$
	$H\mathbb{Z}$	$H\mathbb{Z}[1]$

$[S^1, H\mathbb{Z}[1]]$

$= [S^0, H\mathbb{Z}]$

If X is a based space, Z is a spectrum then $(F(X,Z))_n = F(X, Z_n)$

based maps

(Steenrod: A convenient category of topological spaces, MMJ)

This has a left adjoint: X based space $Z = (Z_n)$ spectrum

$$X \wedge Z = LD$$

$$D_n = X \wedge Z_n.$$

We can also define, for a spectrum $Z = (Z_n)$, $\Omega Z = F(S^1, Z)$

$$\Sigma Z = S^1 \wedge Z.$$

It is not known whether $\Omega Z \overset{?}{\simeq} Z[-1]$, $\Sigma Z \overset{?}{\simeq} Z[1]$.

The "obvious proof" (wrong) that $\Omega Z = Z[-1]$ leads to a diagram

$$\begin{tikzcd}
Z_n \arrow[r] \arrow[d] & \Omega Z_{n+1} \arrow[d] \\
\Omega Z_{n+1} \arrow[r] & \Omega\Omega Z_{n+1}
\end{tikzcd}$$

the diagram fails to commute due to the switch.

Homotopy of spectra : Z, T are spectra :

$$h : [0,1]_+ \wedge Z \to T$$

$$0, 1 : S^0 \to [0,1]_+$$

$S^0 \wedge Z \cong Z$

$S^n, n \in \mathbb{Z}$: $\Sigma^\infty S^m[-k]$ $\quad m - k = n$

HW ① Prove that up to $\cong$ of spectra, the definition of S^n does not depend on m, k.

② Prove that Σ^∞ (as defined) is left adjoint to Ω^∞.

(Due Mon 11/28.)

We define, for $n \in \mathbb{Z}$ and a spectrum Z,

homotopy classes of morphisms

$$\pi_n Z := [S^n, Z]$$

(We have a category hSp Spectra, homotopy classes of morphisms)

spectral sphere

automatically abelian groups

$\pi_m(Z_k)$ $m - k = n$.

An <u>equivalence of spectra</u> is a morphism $f : Z \to T$ which induces an $\cong$ on π_n for all $n \in \mathbb{Z}$.

Using a spectral version of HELP:

<u>Theorem</u>: hSp has co-localization (with respect to equivalence) <u>with</u> respect to a certain class of <u>cell spectra</u>.

Therefore, the derived category DSp exists (the stable homotopy category).

<u>Theorem</u>: Ω preserves equivalences, $D\Omega$, $L\Sigma : DSp \to DSp$ are equivalences of categories.

total derived functor