MATH 695 11/28/2022

Spectra: $Z = (Z_n)$ sequence of based spaces

$$\rho_n : Z_n \xrightarrow{\cong} \Omega Z_{n+1}$$

Morphisms $Z \to Z'$ are sequences of morphisms $Z_n \to Z_n'$ preserving ρ_n.

Prespectra = defined the same way (variant: Prespectra$_Q$ for a cofinal subset $Q \subseteq \mathbb{Z}$)

except $\rho_{(n)} : Z_n \to \Omega^{n'-n} Z_{n'}$

where n' is the successor of n in Q. (No homeomorphism requirement.)

$U_{\mathbb{Q}}$: Spectra $\longrightarrow$ Prespectra$_{\mathbb{Q}}$

has a left adjoint $L = L_{\mathbb{Q}}$ (Freyd-Kelly) <u>called spectrification</u>

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Cell spectra: Z is a <u>cell spectrum</u> when

$$Z = \underrightarrow{\mathrm{colim}}\, Z_{(n)}$$

$(*)_n = *$ (0 object)

$$Z_{(-1)} \to Z_{(0)} \to Z_{(1)} \to \cdots\cdots$$

$Z_{(-1)} = *$, $I_{(n)}$ (set of cells attached in step n) $d_{(n)} : I_{(n)} \to \mathbb{Z}$ (dimension function)

coproduct

Attaching map $f_{(n)} : \bigvee_{i \in I_{(n)}} S^{d_{(n)}(i)-1} \longrightarrow Z_{(n-1)}$. We have

mapping cone

$$\boxed{Z_{(n)} = C f_{(n)}}$$

spectral sphere

$$S^n = \Sigma^\infty S^m [-k]$$

space sphere

$m \geq 0$

$m - k = n$

HW $\Rightarrow$ $\Sigma^\infty S^m[-1] = \Sigma^\infty S^{m-1}$, $m > 0$

More generally, X based space $\Rightarrow$ $(\Sigma^\infty \Sigma X)[-1] \cong \Sigma^\infty X$.

Recall <u>homotopy of spectra</u> $[0,1]_+ \wedge Z \xrightarrow{h} Z'$

$S^0 \wedge Z \cong Z$. } equivalence relation of homotopy $\simeq$ between morphisms $Z \to Z'$. Compatible under composition $\therefore$ the homotopy category hSpectra.

Define for a spectrum Z,

spectral sphere

$$\underset{\text{abelian groups}}{\pi_n Z} := \mathrm{Mor}_{h\mathrm{Spectra}}(\underset{\text{double suspension}}{S^n}, Z) \qquad n \in \mathbb{Z}.$$

<u>Equivalence of spectra</u> $\sim$ is a morphism $\boxed{f : Z \to Z' \text{ which induces } \cong \text{ on } \pi_n \text{ for all } n \in \mathbb{Z}}$.

The derived category of spectra DSpectra with respect to $\sim$ is the stable homotopy category.

Analogs with spaces and chain complexes give HELP, and a Whitehead theorem.

Theorem: hSpectra has colocalization (with respect to $\sim$) in the class of cell spectra. $\square$

Theorem (Lewis, May): In DSpectra, $L\Sigma$, $D\Omega$: DSpectra $\to$ DSpectra are inverse equivalences of categories. $\square$

Ω preserves $\sim$

Note: Given the fact that $\Omega \neq [-1]$ on Spectra, this is where we can declare success of our construction.

But do we have any examples? Recall that we have a couple of very nice pre-spectra:

$HA \qquad HA_n = K(A,n) \quad n \geq 0$

$\qquad\qquad\qquad\ \ \ * \qquad\quad n < 0$

abelian group

$K \qquad K_{2n} = BU \times \mathbb{Z}$

$\qquad\ \ K_{2n+1} = U$

perhaps only ~

$$\boxed{\begin{array}{c} HA_n \xrightarrow{\sim} \Omega HA_{n+1} \\[2em] K_k \xrightarrow{\sim} \Omega K_{k+1} \end{array}}$$

Good enough to do generalised cohomology. But can we get <u>homeomorphisms</u>?

← need for May spectra.

A general problem: Suppose we have a pre-spectrum $Z = (Z_n)$ where

$$\rho_n : Z_n \xrightarrow{\sim} \Omega Z_{n+1}.$$

Can we produce a $\boxed{\text{Map spectrum}}$ $\overline{Z}$ together with a level-wise weak equivalence $Z \to \overline{Z}$.

Idea 1: Take spectrification L.

That does not work. A more refined idea is to take "the left derived functor of spectrification". But we would have to be precise about what this means.

Take-away: We need some case where we have control over the spectrification.

An inclusion pre-spectrum $Z = (Z_n)$ $\sigma_n : Z_n \to \Omega Z_{n+1}$ is a prespectrum where σ_n are inclusions (injective and the topology on the subspace is the subspace topology (made compactly generated)).

Then: $LZ = (T_n)$ where $\boxed{T_n = \operatorname{colim} \Omega^k Z_{n+k}}$

$$\underbrace{\Omega^k Z_{n+k} \to \Omega^{k+1} Z_{n+k+1}}_{\Omega^k \sigma_{n+k}}.$$

For a prespectrum E, the "left derived functor" of spectrification is computed by replacing E with an inclusion prespectrum (using mapping cylinders) and then take spectrification.

HW: ① Suppose $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n$ are topological spaces where $X_n \subseteq X_{n+1}$ has the subspace topology and let $X = \bigcup X_n$ with the union topology. Further assume X_n are T1 (points are closed). Let K a compact topological space and let $f : K \to X$ be a continuous map. Prove that $\mathrm{Im}(f) \subseteq X_n$ for some n.