MATH 695 11/30/2022

Generalized homology and cohomology of based CW-complexes X (extend to all spaces to preserve weak equivalence)

Let E be a (cell) spectrum. One defines

? analogy with Tor, Ext

$$\tilde{E}_n X := \pi_n(E \wedge X)$$

smash level-wise (getting an inclusion prespectrum) and spectrify

$$\tilde{E}^n X := \pi_{-n}(F(X, E))$$

level-wise, already a spectrum

<u>Theorem</u>: All generalized homology and cohomology theories arise in this way from spectra.

What is missing from the picture is an internal $\wedge$ of spectra.
(an analog of the $\otimes$ of chain complexes).

leads to multiplicative algebra (commutative, associative, Lie ...)

Can we define this well enough to have analogues of the algebra?

brave new algebra (Waldhausen)
spectral algebra
higher algebra } Lurie

$\mathbb{Z} \rightsquigarrow S$ (= spectral S^0, the sphere spectrum)

The most basic question is: Do we have a symmetric monoidal $\wedge$ (spectral analog of $\otimes$) on D Spectra? Yes (Adams: Stable homotopy & generalized cohomology III) - LNM 1213 May & al.

needs to be defined

"∞-symmetric monoidal" $\wedge$-product on the ∞-category of spectra? Yes $\leftarrow$ one approach is to modify the category of spectra to have a symmetric monoidal $\wedge$-product and preserve as many as possible formal properties

S-modules: EKMM

At least let us construct the symmetric monoidal $\wedge$ on DSpectra: E, E' spectra

Incorrect idea: $(E \wedge E')_n = E_n \wedge E'_n$ (then maybe be duality)

analogous to $(C \otimes D)_n = C_n \otimes D_n$ for chain cxes.

We want $\Sigma^\infty X \wedge E \sim X \wedge E$

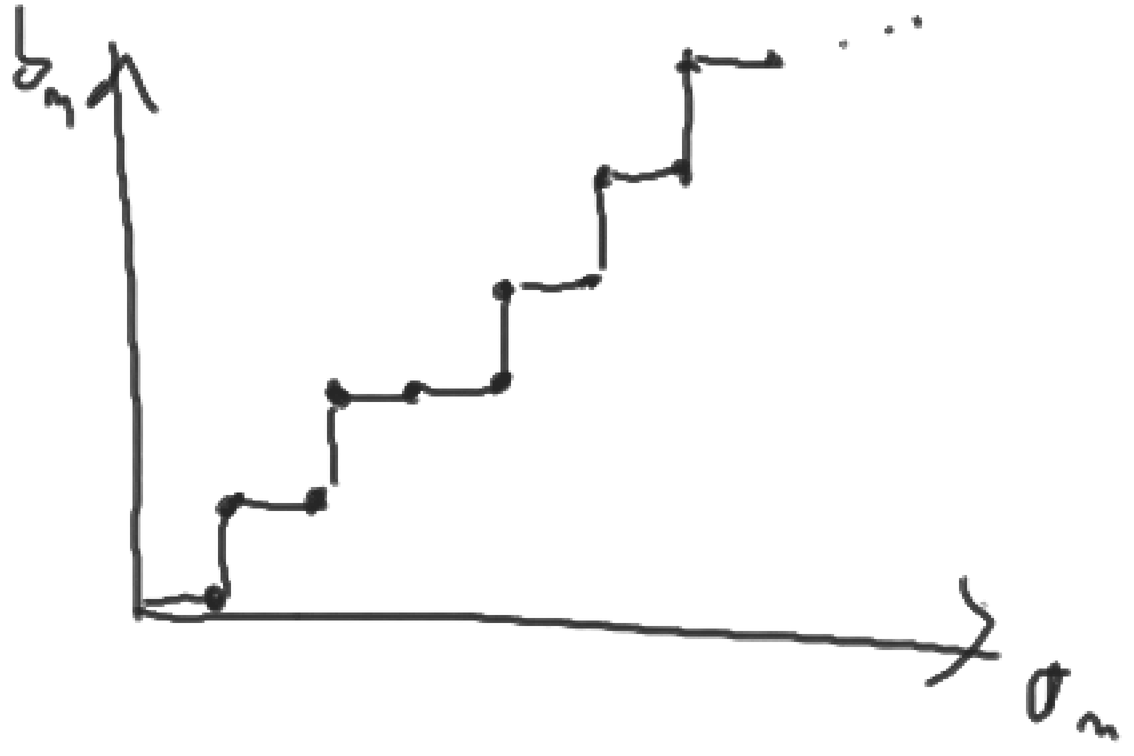
$$\underbrace{\Sigma^\infty \Sigma X}_{\Sigma^\infty X[1] \wedge E} \sim \underbrace{\Sigma X \wedge E}_{X \wedge E[1]} \sim \Sigma^\infty X \wedge E\,[1]$$

Any spectrum is a colimit of shifts of suspension spectra: $E = \operatorname{colim}_n \Sigma^\infty E_n[-n]$

And $\wedge$ should preserve colimits, so we should have

$$(\Sigma E)\wedge E' \sim \Sigma(E\wedge E') \quad , \quad E\wedge \Sigma E' \sim \Sigma(E\wedge E').$$

$$\therefore \quad (\Sigma E)\wedge(\Sigma E') \sim \Sigma^2(E\wedge E').$$



$\therefore \ a_n + b_n = n$

<u>Data</u>: (a,b)

A pair of sequences (a_n), (b_n) of elements of $\{0,1,2,\dots\}$

$a_0 = b_0 = 0$

$$\left.\begin{array}{l} a_n = a_{n-1}+1 \text{ and } b_n = b_{n-1} \\ \text{or} \\ a_n = a_{n-1} \text{ and } b_n = b_{n-1}+1 \end{array}\right\} \forall n \in \mathbb{N}_0$$

$\lim a_n = \lim b_n = +\infty$.

Define for spectra $E = (E_n)$, $E' = (E'_n)$

$$E \wedge_{(a,b)} E' := \text{spectrification of } D_n$$

where $D_n = E_{a_n} \wedge E'_{b_n}$.

(Depending on which sequence steps, $\rho_n : D_n \to \Omega D_{n+1}$ adjoint to $\Sigma D_n \to D_{n+1}$ comes from the E or E' coordinate.)

__Theorem__: The left derived functors of $\wedge_{(a,b)}$ are all equivalent in DSpectra, giving a symmetric monoidal operation on DSpectra.

LNM 1213

Moreover, in DSpectra, $E \wedge ?$ is left adjoint to a functor $F(E, ?)$.

So now we can define for (all) spectra E, X:

$$E_n X = \pi_n (E \wedge X) = (X_n E)$$

analogous to the symmetry of Tor

$$E^n X = \pi_{-n} (F(X, E))$$

One defines a (commutative associative unital) ring spectrum as an object of DSpectra which has morphisms

$$\mu : E \wedge E \to E \qquad \eta : S \to E$$

and we have commutative diagrams

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mu \wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id} \wedge \mu"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"'] & E \end{tikzcd} \qquad \begin{tikzcd} E \wedge E \arrow[r, "\sim"] \arrow[d, "\mathrm{twist}"'] & E \\ E \wedge E \arrow[ur, "\mu"'] & \end{tikzcd} \qquad \begin{tikzcd} S \wedge E \arrow[d, "\eta \wedge \mathrm{Id}"'] & E \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\mathrm{Id}"] & E \wedge S \arrow[d, "\mathrm{Id} \wedge \eta"] \\ E \wedge E \arrow[r, "\mu"'] & E & E \wedge E \arrow[l, "\mu"] \end{tikzcd}$$

commute in D spectra.

Then for an unbased space X, define $E_\wedge X = \tilde{E}_\wedge X_+$

$E_\wedge \Sigma^\infty X =$
$= E_\wedge X$

$E^n \Sigma^\infty X =$
$= E^n X$

If E is a ring spectrum, then $E^* X$ for an unbased space X is a graded-commutative ring.

$$E^n X \otimes E^p X \xrightarrow{?} E^{n+p} X$$

$$\begin{tikzcd}
E^n X & E^p X \\
\Sigma^\infty X_+ \arrow[r, "f"'] \arrow[u] & \Sigma^n E \\
\Sigma^\infty X_+ \arrow[r, "g"] \arrow[uu, bend right] & \Sigma^p E
\end{tikzcd}$$

$$\Sigma^\infty X_+ \xrightarrow[\Sigma^\infty \Delta_+]{} \Sigma^\infty \underbrace{X_+ \wedge X_+}_{(X \times X)_+} \simeq \Sigma^\infty X_+ \wedge \Sigma^\infty X_+ \xrightarrow{f \wedge g} \Sigma^n E \wedge \Sigma^p E$$

uses spaces

$$\begin{tikzcd}
\Sigma^n E \wedge \Sigma^p E \arrow[d, "\wr"] \\
\Sigma^{n+p} E \wedge E \arrow[d, "\Sigma^{n+p} \mu"] \\
\Sigma^{n+p} E
\end{tikzcd}$$

uses ring spectrum

Next time: duality in D Spectra, HR (R ring), $\underline{\underline{K}}$

$\rightarrow$ deriving spectra.

HW 2 Prove that the map $S^2 \to S^2$ given by switching coordinates has degree -1.

(This is where the signs come from in algebraic topology.)