MATH 695 12/2/2022

Duality in spectra

If X is a finite CW-complex (has finitely many cells)

then $\Sigma^\infty X_+$ is strongly dualizable and if $X_+ \subset S^n$

(Hatcher for ANR properties of CW-cxes)

then

$$\underset{F(\Sigma^\infty X_+, \underset{S^0}{S})}{\overset{\parallel}{D\Sigma^\infty X_+}} = \Sigma^\infty C(\underbrace{S^n \setminus X \hookrightarrow S^n}_{\text{"}(S^n,\, S^n \setminus X)\text{"}})[-n]$$

Corollary: $E^k X \cong E_{n-k}(S^n, S^n \setminus X)$

$E_k X \cong E^{n-k}(S^n, S^n \setminus X)$

for any spectrum E.

Even more generally

(HW3) Prove that if X is a non-empty finite CW-cx, $X_+ \subset S^n$ then

$$D\Sigma^\infty X = \Sigma^\infty S^n \setminus X\,[1-n]$$

[Duality preserves cofibration sequences.]

$S^n \setminus X \xrightarrow{j} S^n \to Cj$

$S^0 \subset S^n$ | $S^n \setminus S^0 \simeq S^{n-1}$

Duality on manifolds. When E is a ring spectrum and M is a compact smooth n-manifold we say that M is <u>E-orientable</u> if the tangent bundle (equivalently the normal bundle) is E-orientable (i.e. there is a Thom class in the reduced E-cohomology of the Thom space which restricts to a unit on each fiber). Then we have Poincaré duality

$$E_k M \cong E^{n-k} M, \qquad k \in \mathbb{Z}$$

$\left(D\Sigma^\infty M_+ = \Sigma^\infty M^\nu [-N] \right.$ when $M_+ \subset S^N$, ν is the normal bundle$\left. \right)$

Use the Thom isomorphism.

Example: A (compact) smooth manifold M is orientable with respect to K (K-theory) when M is Spin^c.

Some explanation: $\underset{\text{orthogonal group}}{O(n)} \supset \underset{\det = 1}{SO(n)}$

$\pi_1 SO(n) = \mathbb{Z}/2$ (for $n > 2$). This invariant is called spin, generator:

The universal cover is $\mathrm{Spin}(n)$. Also a compact Lie group. $\mathbb{Z}/2 \subset \underset{\text{center}}{Z}\,\mathrm{Spin}(n)$

$$\mathrm{Spin}^c(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}/2} S^1 \longrightarrow SO(2n)$$

For a manifold M to be Spin^c means that the normal bundle has a $\mathrm{Spin}^c(N)$-structure for some N.

"Structure" on a $\underbrace{\text{vector bundle}}_{\text{locally constant vector space } / X}$ (Assume dim. constant n)

$h_{U,U'} : U \cap U' \to GL_n(\mathbb{R}) \leftarrow$ transition function between two trivializations

coordinate neighborhoods
trivialize the bundle on each of them

satisfy a cocycle condition:

$$h_{U,U'} \circ h_{U',U''} = h_{U,U''}$$

This suggests a kind of cohomology theory

coboundary $f : U \longmapsto \varphi_U$ $\quad \delta f_{U,U'} = f_U \circ f_{U'}^{-1} \ldots$

discuss refinements

here $G = GL_n(\mathbb{R})$

X paracompact, Euclidean metric
can use $O_n(\mathbb{R})$

$\rightsquigarrow$ Čech cohomology group $\check{H}^1_{\text{cont}}(X; G)$

For any group $G \to Gl_n\mathbb{R}$ we say the bundle ξ has a G-structure when the corresponding element of $\underbrace{\check{H}^1_{cont}(X; Gl_n\mathbb{R})}_{\text{determine the bundle up to } \cong}$ lifts to $\underbrace{\check{H}^1_{cont}(X; G)}_{\text{principal } G\text{-bundle}}$.

Husemoller: Fiber bundles

Spin^c-structure on a manifold M $\Rightarrow$ orientability w.r. to K-theory

Spin-structure $\Rightarrow$ orientability w.r. to KO-theory.

existence of the Dirac operator

Poincaré duality map is given by the index.

$\boxed{\text{Monday: What does K-theory mean?}}$

A teaser for studying K-theory a little bit more.

Atiyah-Singer index theorem relates the index of operators to K-theory.