MATH 695 12/5/2022

$BU(n) = \{n\text{-dimensional } \mathbb{C}\text{- vector subspaces of } \bigoplus_{n=1}^{\infty} \mathbb{C}\}$ $\overset{!!}{=} \mathbb{C}^{\infty}$

$\mathbb{C}^{\infty} \cong \mathbb{C}^{\infty} \oplus \mathbb{C}$

$BU(n) \hookrightarrow BU(n+1)$

$BU = \bigcup_{n=1}^{\infty} BU(n)$

Recall: $\Omega^2(BU \times \mathbb{Z}) \simeq BU \times \mathbb{Z}$

$\Omega(BU \times \mathbb{Z}) \simeq U$

X CW-cx:

$K^{2n}X = [X, BU \times \mathbb{Z}]_{\text{unbased}}$

$K^{2n+1}X = [X, U]_{\text{unbased}}$

$U(n) = \{n \times n \text{ matrices } A \mid A\bar{A}^T = I\}$

$U(n) \hookrightarrow U(n+1)$

$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$

$U = \bigcup_{n=1}^{\infty} U(n)$

<u>Theorem</u>: $BU(n)$ classifies n-dimensional complex vector bundles:

Let X be a paracompact space, then there is a canonical bijection

$$\left\{ \begin{array}{c} \cong \text{classes of } n\text{-dim } \mathbb{C} \text{ vector} \\ \text{bundles on } X \end{array} \right\} \xleftarrow{\ \cong\ } [X, BU(n)]_{\text{unbased}}.$$

There is a "tautological" n-dim. cx. vector bundle $\gamma^n_{\mathbb{C}}$ on $BU(n)$:

The fiber over $V \in BU(n)$ is V. More rigorously, the total space is

$$\{ (V, x) \in BU(n) \times \mathbb{C}^\infty \mid x \in V \},$$

$$f^*(\gamma^n) \longleftarrow\!\shortmid \left(f : X \longrightarrow BU(n) \right)$$

One has to prove that this map is well-defined, onto, and injective.

$\underbrace{\text{onto, and injective}}_{\text{Milnor-Stasheff. Characteristic classes}}$

homotopic maps $X \longrightarrow Y$ induce isomorphic bundles.

$h : X \times [0,1] \to Y$ $\qquad Q \xrightarrow{\varphi} X \times [0,1]$

γ is an n-bundle on Y

$$\varphi^{-1}(x,t) = \mathrm{Iso}\left(h_0^*(\gamma)_x , h_t^*(\gamma)_x \right)$$

topology by local triviality.

φ is locally a product $\therefore$ a fibration

Hatcher: a Concise course

we have a section of φ on $X \times \{0\}$. By the HLP, it extends and at $t=1$ it gives us what we want.

(onto)
Existence (for X compact): Define it on coordinate
("Gauss map")
neighborhoods (pick independent targets)
glue via partition of unity.

<u>injectivity</u>: A good concept classifying map $\Leftrightarrow E(\xi) \xrightarrow{f} \mathbb{C}^\infty$

(classifying maps $\to BU(n)$
for the same bundle ξ
on X are homotopic)

total space

linear and injective on fibers

Two such maps f, g are homotopic through maps of the same kind.

"Milnor trick:" homotope f by composing with

$t(x_1, x_2, x_3, \dots) + (1-t)(0, x_1, 0, x_2, 0, x_3, \dots)$

never be 0
(consider the top n with $x_n \neq 0$)

(Gram-Schmidt if desired)

standard $(x_1, 0, x_2, 0, \dots)$ linearly homotope. $\square$

Atiyah: K-theory.

Restrict to X compact (for us, also CW).

A notion of vector bundle on X: a $\mathbb{C}$-vector space over X locally constant and locally finite-dimensional (dimension locally constant)

$\mathbb{C}^n$ $\mathbb{C}^m$ OK

$$\underbrace{\{\text{Vector bundles on } X\}/\cong}_{\substack{\text{commutative monoid under } \oplus \\ \text{(commutative semiring under } \oplus, \otimes)}} \xleftrightarrow{\cong} [X, BU(0) \sqcup BU(1) \sqcup BU(2) \sqcup \cdots]$$

commutative monoid under $\oplus$ (commutative semiring under $\oplus, \otimes$).

Grothendieck construction

$$\begin{tikzcd} \mathrm{Ab} \arrow[r, "\text{forget}"] & \text{Commutative Monoids} \arrow[l, bend left, "K"] \end{tikzcd}$$

left adjoint

$$\begin{tikzcd} \text{Comm. rings} \arrow[r, "\text{forget}"] & \text{Commutative semirings} \arrow[l, bend left, "K"] \end{tikzcd}$$

(Constructing $\mathbb{Z}$ out of $\mathbb{N}_0$.)

$\{(x,y)\}/\sim$ $\qquad (x,y)\sim(z,t) \quad \exists u$

think $x-y$ $\qquad x+t+u = y+z+u$

One defines for X compact (CW)

$$K(X) = K\left(\{\cong \text{ classes of } \overset{\mathbb{C}}{\text{vector}} \text{ bundles on } X\}, \oplus\right)$$

$\uparrow$ add $\otimes$

$K(X)$ becomes a ring.

union with weak topology

$$\operatorname{colim}\left(\left(BU(0) \sqcup BU(1) \sqcup BU(2) \sqcup \cdots\right) \xrightarrow[\hookrightarrow]{\text{Shift}} \left(\overbrace{BU(0) \sqcup BU(1) \sqcup BU(2) \sqcup \cdots}^{BU^{\Box}}\right) \xrightarrow[\hookrightarrow]{\text{shift}} \cdots\right)$$

$$\cong BU \times \mathbb{Z}$$

$BU(n) \hookrightarrow BU(n+1)$

$\mathbb{C}^{n} \cong \mathbb{C}^{n} \oplus \mathbb{C}$

For X compact, a map $X \to BU \times \mathbb{Z}$ lands in the n-th copy of $BU^{\diamond}$ for some n. So we send it to $\xi - n \in K(X)$

if classifies a vector bundle ξ

To go backwards, we recall that any bundle η on X has a complement η' such that $\eta \oplus \eta' \cong N$. (Finite version of existence argument in classification.)

$$\xi - \eta \in K(X) \longmapsto$$ send it to the Nth copy of $BU^{\diamond}$ via the classifying map of $\xi \oplus \eta'$.

$$(\xi \oplus \eta') - (\eta \oplus \eta')$$
$$= (\xi \oplus \eta') - N$$

These two correspondences are inverse to each other. So we proved

Theorem: For X compact CW,

$$K(X) = K^0(X). \qquad \square$$

$K\{\cong \text{classes of vector bundles}\}$ $\qquad$ $[X, BU \times \mathbb{Z}]$ unbased

Note: This also makes $K^0(X)$ into a ring. This can be realised on the homotopy level to prove that K is a(n E_∞)- ring spectrum.