MATH 695 12/7/2022

Recall that $\mathbb{CP}^\infty = BU(1) = \{$ cx. lines through the origin in $\mathbb{C}^\infty\}$

$BU(n) = \{$cx. v. subspaces of $\mathbb{C}^\infty$ of dim $n\}$

$\gamma^1 =$ universal line bundle on $\mathbb{CP}^\infty$

$\gamma^n =$ univ. cx. vector bundle on $BU(n)$

Total space $E(\gamma^n) = \{(x, V) \in \mathbb{C}^\infty \times BU(n) \mid x \in V\}$

We say that a ring spectrum E is <u>complex-oriented</u> if the complex line bundle γ^1 is E-oriented.

($\exists$ Thom class, $\therefore$ Thom isomorphism)

In fact, this implies γ^n (and hence every cx. bundle) is E-oriented

Example: · HR where R is a commutative ring is cx.-oriented

(cx. structure determines an orientation $z_1, \dots z_n$ cx. basis on a v.s. $(z_1, iz_1, z_2, iz_2, \dots z_n, iz_n))$

$$U(n) \subset SO(2n)$$

· K theory is cx.-oriented. $[\gamma^1] \in K^0 \mathbb{C}P^\infty$ (but we need it in dim 2)

$$x = [\gamma^1] - 1 \in \tilde{K}^0 \mathbb{C}P^\infty \underset{\text{Bott}}{\cong} \tilde{K}^2 \mathbb{C}P^\infty = \tilde{K}^2 (\mathbb{C}P^\infty)^{\gamma^1}$$

$\mathbb{C}P^\infty = (\mathbb{C}P^\infty)^{\gamma^1}$ by previous HW

Atiyah-Hirzebruch Spectral sequence (did not discuss for cohomology for an ∞-d. space)

$$K^i \mathbb{C}P^n = \begin{cases} \mathbb{Z}[x]/x^{n+1} & i \text{ even} \\ 0 & \text{else} \end{cases}$$

Mittag-Loeffler condition ⇒

$$K^{i}\mathbb{C}P^{\infty} = \begin{cases} \mathbb{Z}[[x]] & \text{if } i \text{ even} \\ 0 & \text{if } i \text{ odd.} \end{cases}$$

Details: Spectral sequence of rings (usually, cohomological) $d_r(uv) = (d_r u)v + (-1)^{|u|} u d_r(v)$

total degree $p+q$

Independently of evenness, if E is a cx.-oriented spectrum, the AHSS for $\mathbb{C}P^n$ will always have E_2-term $E_*[x]/x^{n+1}$ where x is the cx. orientation (and hence is a permanent cycle)

∴

$$E^*\mathbb{C}P^{\infty} = E^*[[x]].$$

← In Algebraic topology, only consider the homogeneous elements

Similarly, $E^*(\underbrace{\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty}_{n}) = E_*[[x_1, \dots, x_n]]$

cohomological degree 2

homological degrees should get a − $(E_d = E_*(*) = E^{-*}(*))$

when E is cx.-oriented

In fact,

$$\begin{tikzcd}
E^*BU(n) \arrow[d, hook] & = E_*[[c_1, \dots, c_n]] \arrow[d, hook] & c_i \arrow[d, mapsto] \\
E^*(\mathbb{CP}^\infty)^n & = E_*[[x_1, \dots, x_n]] & \sigma_i(x_1, \dots, x_n)
\end{tikzcd}$$

Milnor-Stasheff characteristic classes for ordinary cohomology

$\mathbb{CP}^\infty \hookrightarrow BU(n)$

$\mathbb{CP}^\infty$ has a multiplication (you can make a model which is a topological abelian group)

all we need is comm. assoc. unital up to homotopy.

<u>Construction 1</u> : $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$.

X CW-cx,
$[X, \mathbb{C}P^\infty] = H^2(X;\mathbb{Z})$

$$\begin{tikzcd}
\overset{\gamma_1^1}{\mathbb{C}P^\infty} \times \overset{\gamma_2^1}{\mathbb{C}P^\infty} \arrow[r, "\gamma_1^1 \otimes \gamma_2^1"] & \mathbb{C}P^\infty
\end{tikzcd}$$

The multiplication on $\mathbb{C}P^\infty$ <u>represents</u> the $\otimes$-product of cx. line bundles.

If E is a cx.-oriented spectrum, the multiplication on $\mathbb{C}P^\infty$ induces a map

$$\begin{tikzcd}
E^*[[x]] = E^*\mathbb{C}P^\infty \arrow[r, "E^*\otimes"] & E^*\mathbb{C}P^\infty \times \mathbb{C}P^\infty = E^*[[y,z]] \\
x \arrow[r, mapsto] & F(y,z)
\end{tikzcd}$$

↖ a power series with coefficients in E^{even}

What are the properties of this power series? (consequences of $\otimes$ on $\mathbb{C}P^\infty$ being commutative, associative, unital up to $\cong$)

$$\left.\begin{aligned} F(x,0) &= F(0,x) = x \\ F(x,y) &= F(y,x) \\ F(F(x,y),z) &= F(x,F(y,z)) \end{aligned}\right\}$$

a power series $F(y,z) \in R[[y,z]]$ with these properties is called a <u>formal group law</u> (FGL).

$\nwarrow$ commutative ring

What is the formal group law for the two examples we have, $H\mathbb{Z}$, K?

In $H\mathbb{Z}$, $\quad H\mathbb{Z}^*\mathbb{CP}^\infty = \mathbb{Z}[x] \qquad |x| = 2$

↖ no opportunity for homogeneous terms in the same degree

By the same token, $F(x,y)$ can only have linear terms

$\therefore F(x,y) = x + y \qquad$ (the $\underline{\text{additive}}$ FGL).

↑ the formula for $c_1(\xi \otimes \eta)$ in terms of $c_1(\xi), c_1(\eta)$ where ξ, η are line bundles.

↑ 1st Chern class of $\gamma^1_{\mathbb{C}}$ = cx. orientation of $\gamma^1_{\mathbb{C}}$

In K-theory, $c_1(\xi) = \xi - 1$.

$c_1(\xi \otimes \eta) = \xi\eta - 1$ in terms of $c_1(\xi), c_1(\eta)$

$$" \; (\xi - 1)(\eta - 1) + (\xi - 1) + (\eta - 1)$$

$$F(x,y) = x + y + xy \qquad \left(= (1+x)(1+y) - 1 \right)$$

suppressing the notation for the Bott element to put everything in the same degree.

The *multiplicative* FGL.

Example: $K^* \mathbb{RP}^\infty$

Consider on $\mathbb{CP}^\infty$ the cx. line bundle $\xi = (\gamma^1)^2$. Recall the cofibration sequence

$$S(\xi) \longrightarrow X \longrightarrow X^{\xi}$$

for a vector bundle ξ on X with Euclidean metric ($S(\xi)$ is the unit sphere bundle).

$$\mathbb{RP}^\infty \longrightarrow \mathbb{CP}^\infty \longrightarrow (\mathbb{CP}^\infty)^{(\gamma^1)^2}$$

By cx. orientation, Thom isomorphism in K-theory

In K-theory, we get the Gysin sequence in cohomology

$$\ldots \leftarrow K^* \mathbb{R}P^\infty \longleftarrow K^* \mathbb{C}P^\infty \xleftarrow[{[2]x}]{F(x,x)} K^* \mathbb{C}P^\infty \leftarrow \ldots$$

$$0 \leftarrow \mathbb{Z}[[x]]/(2x+x^2) \longleftarrow \mathbb{Z}[[x]] \xleftarrow[2x+x^2]{} \mathbb{Z}[[x]] \leftarrow 0$$

additively

$$\mathbb{Z} \oplus \mathbb{Z}_2 \quad = \mathbb{Z}[[x]]/x(x+2) = K^* \mathbb{R}P^\infty \quad \Big| \quad H^*(\mathbb{R}P^\infty; \mathbb{Z}) = \mathbb{Z}[x]/(2x)$$

non-torsion

Atiyah-Segal completion theorem.