MATH 695 12/9/2022

If E is a complex-oriented spectrum (complex line bundles are E-oriented $\Rightarrow$ all complex bundles)

then there is a formal group law (FGL) on the ring $R = E^{\text{even}}(*)$

coming from the $\otimes : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$

classifies the $\otimes$ of complex line bundles.

Recall that an FGL on R is a powerseries $F(x,y) = x +_F y \in R[[x,y]]$ which satisfies

$$x +_F 0 = 0 +_F x = x$$
$$x +_F y = y +_F x$$
$$(x +_F y) +_F z = x +_F (y +_F z)$$

Examples: $E = H\mathbb{Z}$ $\qquad x +_F y = x + y$ (additive FGL)

$E = K$ $\qquad x +_F y = x + y + xy$ (multiplicative FGL)

Bott class

What do we know about FGL's?

Lazard: There exists a universal FGL Φ on a ring L.

FGL's for a ring R are in bijective correspondence with homomorphisms of rings $h: L \to R$

$$\Phi(x,y) = \sum a_{ij} x^i y^j \quad \circledast$$

$$F(x,y) = \sum h(a_{ij})\, x^i y^j$$

Proof: $L = \mathbb{Z}[a_{ij}] / $ relations necessary to make $\circledast$ into an FGL e.g. $a_{ij} = a_{ji}$

$\square$

Lazard's theorem: $L = \mathbb{Z}[x_1, x_2, x_3, x_4, \dots]$.

(The x_k is equal to $a_{ij} \cdot C_{ij}^k$ mod higher degree polynomials in x_m.)

constant

Is there a complex-oriented spectrum whose FGL is the universal FGL Φ?

Yes: The complex cobordism spectrum MU.

What is cobordism? Question we are asking is: Is a compact (smooth) manifold the boundary of another manifold?

← unoriented cobordism MO

One can improving cobordism theory by adding "stable normal data".

$\exists N \quad M \overset{?}{=} \partial N$

Is there a normal bundle ν_M with a given structure

- complex — MU_n
- oriented — MSO_n
- spin — $MSpin_n$
- trivial — $M*_n$
- none — MO_n

$\nu_M = \nu_N|_M$

and also a normal bundle ν_N with the same structure which restricts to ν_M with the structure.

$\tau_N|_M = \tau_M \oplus 1$

In fact, cobordism is an equivalence relation: $M_1 \sim M_2$

$M_1 - M_2 = \partial N$

Set of cobordism classes with given normal data is an abelian group

$M_1 + M_2 = M_1 \amalg M_2$

René Thom (1950's) realised that cobordism groups are the homotopy groups of a certain spectrum (called the Thom spectrum)

MU, MSO, MSpin, M*, MO, ...

Construction of MU: Define a prespectrum (D_{2k}) $k \in \mathbb{N}_0$

$$D_{2k} = (BU(k))^{\gamma^k}$$

The Thom space of the universal complex k-bundle

Connecting maps: $\gamma^{k+1}\big|_{BU(k)} = \gamma^k \oplus 1_{\mathbb{C}}$

on $BU(k+1)$

$MU := L(D)$

spectrification

$$BU(k)^{\gamma^k \oplus 1_{\mathbb{C}}} = \Sigma^2 BU(k)^{\gamma^k}$$

$$BU(k)^{\gamma^k \oplus 1_{\mathbb{C}}} \to BU(k+1)^{\gamma^{k+1}}$$

$\rho_{2k}^{2k+2} : D_{2k} \to \Omega^2 D_{2k+2}$ adjoint to this map

Thom's theorem shows that $M_* = \Sigma^\infty S^0 = S$ (the spectral sphere)

framed cobordism

Framed cobordism groups = stable homotopy groups of spheres

Proving Thom's theorem: Milnor-Stasheff (Ch. 17 ?)

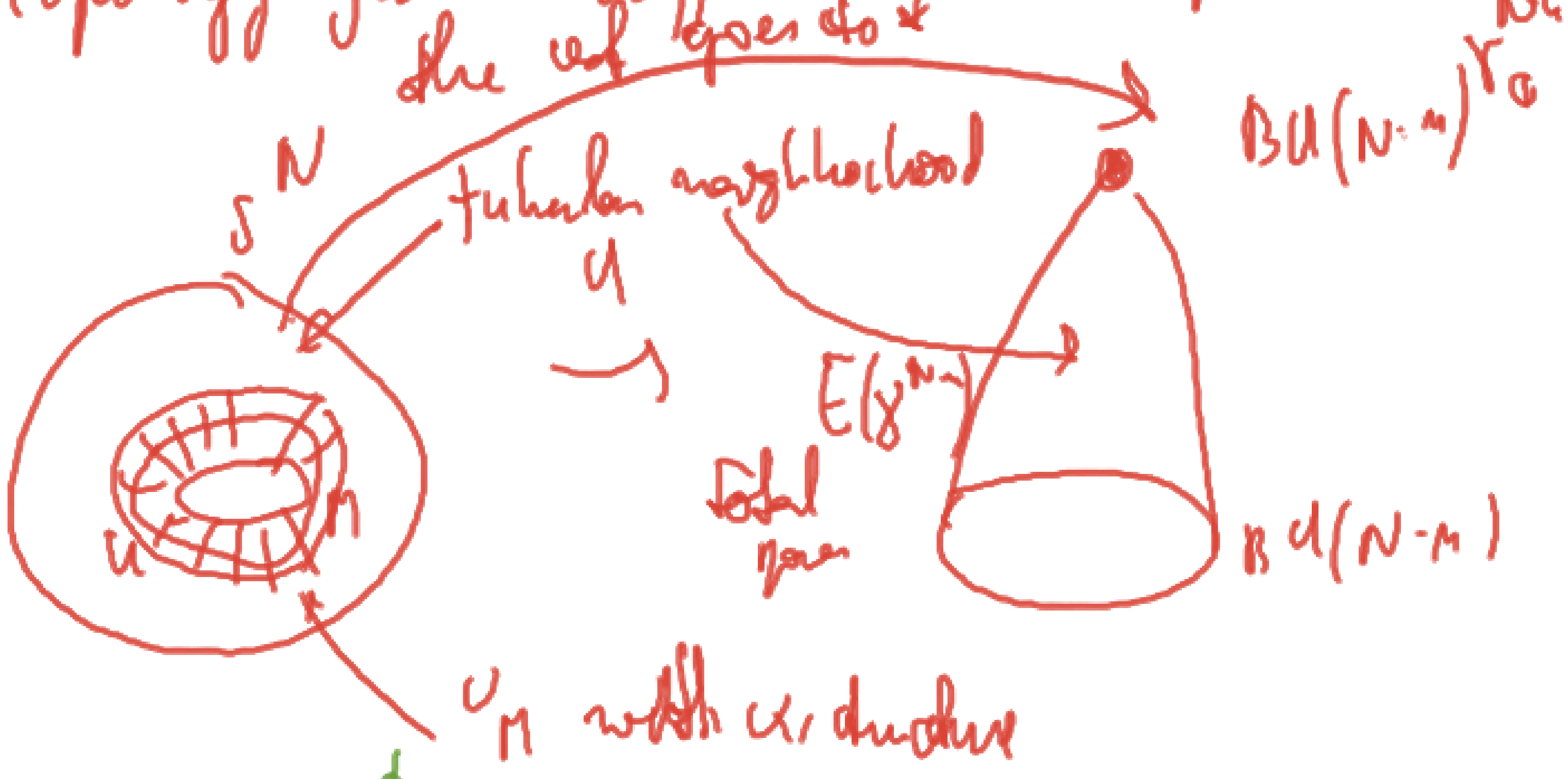
Milnor: Topology from a differentiable viewpoint

$$\pi_N\left(BU(N-n)^{\gamma_{\mathbb{C}}^{N-n}}\right)$$

$$\updownarrow$$

$$\pi_n MU$$

Pontryagin-Thom construction



Reverse process !!

The homotopy theory requires different methods

successful for MU, MO, MSO, $MSpin$

+ a few more

Adams spectral sequence.

Adams: Stable homotopy & generalized cohomology

(Parts I, II proves calc. of MU_*)

Stong: Cobordism