THE HOMOTOPY AND EXCISION AXIOMS FOR SINGULAR HOMOLOGY

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Our approach to the homotopy and excision axioms generally follows [1].

1. The homotopy axiom

Theorem 1. *Two homotopic maps of pairs induce the same map in singular homology.*

Recall that if $f, g : C \to D$ are chain maps, a *chain homotopy* $h : f \simeq g$ consists of homomorphisms of abelian groups $h = h_n : C_n \to D_{n+1}$ such that

$$dh + hd = f - g.$$

Lemma 2. *If X is a contractible space, then $X \to *$ induces an isomorphism in homology. (In particular, $H_0(X) = \mathbb{Z}$, $H_n(X) = 0$ for $n > 0$.)*

Proof. If X is contractible, we have a homotopy $k : X \times [0,1] \to X$ with $k(x,0) = x$, $k(x,1) = *$ for some $* \in X$ (independent of x). Let a chain map $\epsilon : CX \to CX$ (where CX is the singular chain complex of X) be defined on a singular simplex $\sigma : \Delta^n \to X$ by $\epsilon(\sigma) = 0$ for $n > 0$, and $\epsilon(\sigma) = *$ for $n = 0$ (identifying singular 0-simplices with points). We shall exhibit a chain homotopy

$$h : Id_{CX} \simeq \epsilon,$$

which implies the statement (by considering what ϵ induces on homology).

In effect, we may define, for $\sigma : \Delta^n \to X$,

$$h(\sigma) : \Delta^{n+1} \to X$$

on barycentric coordinates by

$$[t_0, \dots, t_{n+1}] \mapsto k(\sigma[\frac{t_1}{1-t_0}, \dots \frac{t_n}{1-t_0}], t_0)$$

for $0 \le t_0 < 1$, and

$$[0, \dots, 0, 1] \mapsto *.$$

One sees that
$$dh(\sigma) = \sigma - \sum_{i=0}^{n} (-1)^i h(\sigma \circ \partial_i)$$
except for $n = 0$, where
$$dh(\sigma) = \sigma - *,$$
(identifying singular 0-simplices with points). Thus, h is the required chain homotopy.

□

Lemma 3. *Let $\iota_t : X \to X \times [0,1]$ be given by $x \mapsto (x,t)$. Then there is a natural chain homotopy $h : C(\iota_0) \simeq C(\iota_1)$. Naturality means that for a continuous map $f : X \to Y$, the following diagram commutes:*

$$\begin{tikzcd} C_n(X) \arrow[r, "h_n"] \arrow[d, "f_*"'] & C_{n+1}(X \times [0,1]) \arrow[d, "(f\times[0,1])_*"] \\ C_n(Y) \arrow[r, "h_n"] & C_{n+1}(Y \times [0,1]). \end{tikzcd} \tag{1}$$

Proof. We shall construct h_n simultaneously for all topological spaces X by induction on n. For $n = 0$, choose an affine map $\iota : \Delta^1 \to [0,1]$ which sends $(0,1) \mapsto 0$, $(1,0) \mapsto 1$. Then for a singular 0-simplex in X, which can be identified with a point $x \in X$, let
$$h(x) = const_x \times \iota.$$
Now suppose h_{n-1} is defined and natural in the sense of (1) (with n replaced by $n-1$). We will first define $h(\kappa_n)$ where $\kappa_n = Id : \Delta^n \to \Delta^n$. Then for any singular n-simplex $\sigma : \Delta^n \to X$ for any topological space X, we have
$$\sigma = \sigma_*(\kappa_n).$$
Thus, to satisfy naturality (1), we can (and must) put
$$h(\sigma) = \sigma_*(h(\kappa_n)).$$
Now to construct $\lambda = h(\kappa_n)$, we must solve the equation
$$d\lambda + h(d\kappa_n) = (\iota_0)_*\kappa_n - (\iota_1)_*\kappa_n,$$
or
$$d\lambda = (\iota_0)_*\kappa_n - (\iota_1)_*\kappa_n - h(d\kappa_n). \tag{2}$$
Let us verify that the right hand side of (2) is a cycle. We have
$$\begin{aligned} &d((\iota_0)_*\kappa_n - (\iota_1)_*\kappa_n - h(d\kappa_n)) = \\ &(\iota_0)_*d\kappa_n - (\iota_1)_*\kappa_n - dhd\kappa_n = \\ &dhd\kappa_n - hdd\kappa_n - dhd\kappa_n = 0, \end{aligned} \tag{3}$$

as required. Now for $n > 0$, the right hand side of (2) being a cycle, it is also a boundary, since $\Delta^n \times [0,1]$ is contractible, and thus,

$$H_n(\Delta^n \times [0,1]) = 0$$

by Lemma 2. Thus, λ exists, and the induction step is complete.

□

Comments: 1. The method used in this proof is known as the *method of acyclic models*. In [1], a general categorical version of this method is treated.

2. The above proof demonstrates vividly the difference between "natural" and "canonical": The homotopy constructed is natural in the sense of (1), but is by no means canonical, in that we have no preferred choice for the class λ in the induction step.

Now Theorem 1 follows from Lemma 3: A homotopy of pairs is a morphism of pairs of the form $k : (X_1 \times [0,1], X_2 \times [0,1]) \to (Y_1, Y_2)$. The chain homotopy of Lemma 3 gives a chain homotopy between the chain maps

$$(X_1, X_2) \to (X_1 \times [0,1], X_2 \times [0,1])$$

given by

$$(x_1, x_2) \mapsto ((x_1, t), (x_2, t))$$

with $t = 0, 1$. Compose this with

$$k_* : C((X_1 \times [0,1], X_2 \times [0,1])) \to C(Y_1, Y_2),$$

and take homology.

2. The excision axiom

Theorem 4. *Let Z and Y be subsets of a topological space X such that the closure of Z in X is contained in the interior of Y in X. Then the inclusion of pairs $(X \smallsetminus Z, Y \smallsetminus Z) \to (X, Y)$ induces an isomorphims on singular homology.*

We shall prove the following

Proposition 5. *Let $\mathcal{U}$ be a set of subsets of a topological space X whose interiors cover X, and let $C^{\mathcal{U}}$ be the chain subcomplex of $C(X)$ generated by those singular simplices whose image is contained in one of the sets in $\mathcal{U}$. Then*

$$C^{\mathcal{U}}(X) \to C(X)$$

induces an isomorphism in homology.

Using the Proposition, we immediately have a

Proof of Theorem 4. Let $\mathcal{U} = \{Y, X \smallsetminus Z\}$. We have a diagram of chain complexes where the rows are induced by inclusions, and exact:

$$\begin{tikzcd}
0 \arrow[r] & C(Y) \arrow[r] \arrow[d, "Id"'] & C^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\subseteq"'] & C(X \smallsetminus Z, Y \smallsetminus Z) \arrow[r] \arrow[d, "\subseteq"] & 0 \\
0 \arrow[r] & C(Y) \arrow[r] & C(X) \arrow[r] & C(X,Y) \arrow[r] & 0.
\end{tikzcd}$$

Apply homology, and then use Proposition 5 and the 5-lemma. □

The first step toward proving Proposition 5 is defining a natural *barycentric subdivision* chain map

$$sd : CX \to CX.$$

Let

$$\alpha : \{0, \dots, n\} \to \{0, \dots, n\} \tag{4}$$

be a permutation. We will define a singular n-simplex

$$\lambda_\alpha : \Delta^n \to \Delta^n.$$

First define λ_{Id} as the affine map which has

$$\lambda_{Id}([1, 0, \dots, 0] = [\frac{1}{n+1}, \frac{1}{n+1}, \dots \frac{1}{n+1}],$$

$$\lambda_{Id}([0, 1, 0, \dots, 0] = [0, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}],$$

$$\dots$$

$$\lambda_{Id}([0, 0, \dots, 1] = [0, \dots, 0, 1],$$

Then define

$$\lambda_\alpha([t_0, \dots, t_n]) = \lambda_{Id}([t_{\alpha^{-1}(0)}, \dots, t_{\alpha^{-1}(n)}]).$$

Finally, put, for a singular n-simplex $\sigma : \Delta^n \to X$,

$$sd(\sigma) = \sum_\alpha sign(\alpha) \sigma \circ \lambda_\alpha \tag{5}$$

where the sum is over all permutations (4). To see that sd is a chain map, taking the 0'th fact on the right hand side of (5) is $sd(d\sigma)$, the remaining terms cancel in pairs, taking the i'th face for two permutations σ, $\tau \circ \sigma$ where τ is the 2-cycle permutation $(i-1, i)$.

Lemma 6. *There exists a natural chain homotopy*

$$h : sd \simeq Id.$$

Proof. We use the method of acyclic models. We have $sd_0 = Id_0$, so we can put $h_0 = 0$. Now suppose that h_{n-1} is constructed. We shall construct

$$h_n(\kappa_n)$$

where $\kappa_n = Id : \Delta^n \to \Delta^n$. Then we can, again, represent any singular n-simplex $\sigma : \Delta^n \to X$ as $\sigma_* \kappa_n$, and thus we can (and must) put

$$h_n(\sigma) = \sigma_* h_n(\kappa_n).$$

To find $\lambda = h_n(\kappa_n)$, we have, again, the equation

$$d\lambda = sd(\kappa_n) - \kappa_n - h_{n-1}(d\kappa_n).$$

We find that the right hand side is a cycle in $C_n X$ by a calculation identical to (3). Thus, it is a boundary by Lemma 2. Thus, we can solve for λ, completing the induction step. □

Proof of Proposition 5. Consider the short exact sequence of chain complexes

$$0 \to C_{\mathcal{U}}(X) \to C(X) \to C(X)/C^{\mathcal{U}}(X) \to 0.$$

By the long exact sequence in homology, it suffices to show that the last term has homology 0. A cycle in $C(X)/C^{\mathcal{U}}(X)$ is represented by a chain $c \in C(X)$ such that

$$d(c) \in C^{\mathcal{U}}(X). \tag{6}$$

By the Lebesgue number theorem, however, there exists an $n \in \mathbb{N}$ such that $sd^n(c) \in C^{\mathcal{U}}(X)$. Now by Lemma 6 (and induction), there exists a chain homotopy

$$k : sd^n \simeq Id,$$

i.e.

$$dk(c) + k(dc) = sd^n(c) - c,$$

or

$$c + dk(c) = sd^n(c) - k(dc).$$

Observe that by (6), the right hand side is in $C^{\mathcal{U}}(X)$. Thus,

$$c \in Im(d) + C^{\mathcal{U}}(X),$$

or, in other words, c is a boundary in $C(X)/C^{\mathcal{U}}(X)$, as required. □

References

[1] J.Munkres: Elements of Algebraic Topology, ISBN 0201627280