MATH 417 1/09/2023

unit matrix

$$I = I_m = \underbrace{\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{pmatrix}}_{m \times m}$$

unit matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

not a unit matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 6 & 4 \end{pmatrix}$$

Matrix multiplication is associative

$(AB)C = A(BC)$

$A = (a_{ij}) \quad B = (b_{jk})$

$C = (c_{kl})$

$= (x_{il})$

$x_{il} = \sum_{j,k} a_{ij} b_{jk} c_{kl}$

Question: What about inverse matrix?

$AA^{-1} = A^{-1}A = I \quad$ Does it exist?

Not ~~always~~: Example: zero matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

you always get the 0 matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so the 0 matrix has no inverse.

Systems of linear equations

Examples: Probably, we can solve a system of two linear equations.

Assuming we know this.

$$2x + y = 5$$
$$3x + 2y = 4$$

A pain to do if we have a lot of variables!

One method of solution:

$$y = 5 - 2x$$

Plug into the same equation:

$$3x + 2(5 - 2x) = 4$$
$$3x + 10 - 4x = 4$$
$$-x = -6$$

$$\boxed{\begin{aligned} x &= 6 \\ y &= -7 \end{aligned}}$$

Check: $12 - 7 = 5$
$18 - 14 = 4$ ✓

Reason to study further:

① The improvised method becomes difficult with more equations and variables

② There are some funny cases. Do we understand them?

$$\boxed{\begin{aligned} x + 2y &= 5 \\ 2x + 4y &= 9 \end{aligned}}$$

$2x + 4y = 10$

$9 = 10$??

This system of equations does not have a solution.

Another example:

$$\begin{array}{r} x + 2y = 5 \\ 2x + 4y = 10 \\ \hline \end{array}$$

← no contradiction, but the second equation follows from the first one (and vice versa).

So the only equation we have is

$$x + 2y = 5$$

What does it mean to "solve" this?

$\boxed{\text{Underdetermined} \Rightarrow \text{cannot solve}}$ ← not a definitive answer.

Somebody could ask: give me at least one solution.

$x = 5$, $y = 0$. | We got this answer by plugging in 0 for y and then figuring x.

$$x + 2y = 5$$

We can plug in *anything* for y. So y becomes a *free parameter*.

$$\boxed{\begin{aligned} y &= A \\ x &= 5 - 2A \end{aligned}}$$

in some sense, this is a complete solution of the underdetermined equation:

note alternative solution:

$$\boxed{\begin{aligned} x &= B \\ y &= \frac{5-B}{2} \end{aligned}}$$

We introduce a list of *free parameters*, and express all the unknowns in terms of them.

(This is useful in geometry – converting equations for lines, planes, etc. into *parametrizations*.)

These are the only scenarios that can happen.

Our job is to figure out how to do it for many variables and many equations. (What happens to the geometry? we will learn how to do it in any dimension.)

It helps to use matrices in solving linear equations

$$2x + y = 5$$
$$3x + 2y = 4$$

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 3 & 2 & 4 \end{array}\right) \quad -1$$

means:

$$2x + y = 5$$
$$3x + 2y = 4$$

-1 subtract first equation from second

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copy the first row

$$\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & 1 & -1 \end{array}\right)$$

switch rows

$$2x + y = 5$$
$$x + y = -1$$

copy so I don't lose it

switch equations

subtract first row from second

Strategy: get a "1" to the upper left corner = A <u>PIVOT</u>

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 5 \end{array}\right)$$

$$x + y = -1$$
$$2x + y = 5$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 1 & 5 \end{array}\right) \; -2 \qquad \qquad x + y = -1, \quad 2x + y = 5 \; -2$$

A pivot

Next: clear all elements (above and) below the pivot

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -1 & 7 \end{array}\right) \; \cdot(-1) \qquad \qquad x + y = -1, \quad -y = 7 \; \cdot(-1)$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & \boxed{1} & \boxed{-7} \end{array}\right) \; -1 \qquad \qquad x + y = -1, \quad y = -7 \; -1$$

the second pivot

(the row and column where the first pivot is "doesn't count")

$$\left(\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -7 \end{array}\right) \qquad \qquad x = 6, \quad y = -7$$

What we just learned is called

GAUSS ELIMINATION

a German mathematician active around 1800

HW due, along with all the other HW assigned this week, WEDNESDAY 1/18 10AM (no class next Monday). On Gradescope on Friday

1. Solve in parametric form:

$$2x + 3y = 7$$

(2) Solve using Gauss elimination (must be used for full credit):

$$x + 5y = 10$$
$$2x + 9y = 1$$