MATH 417 1/11/23

Solving systems of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

matrix form

$$\leadsto \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Elementary row operations

Symbol

1. Add a multiple of one row to another row λ
   (don't change the original row)

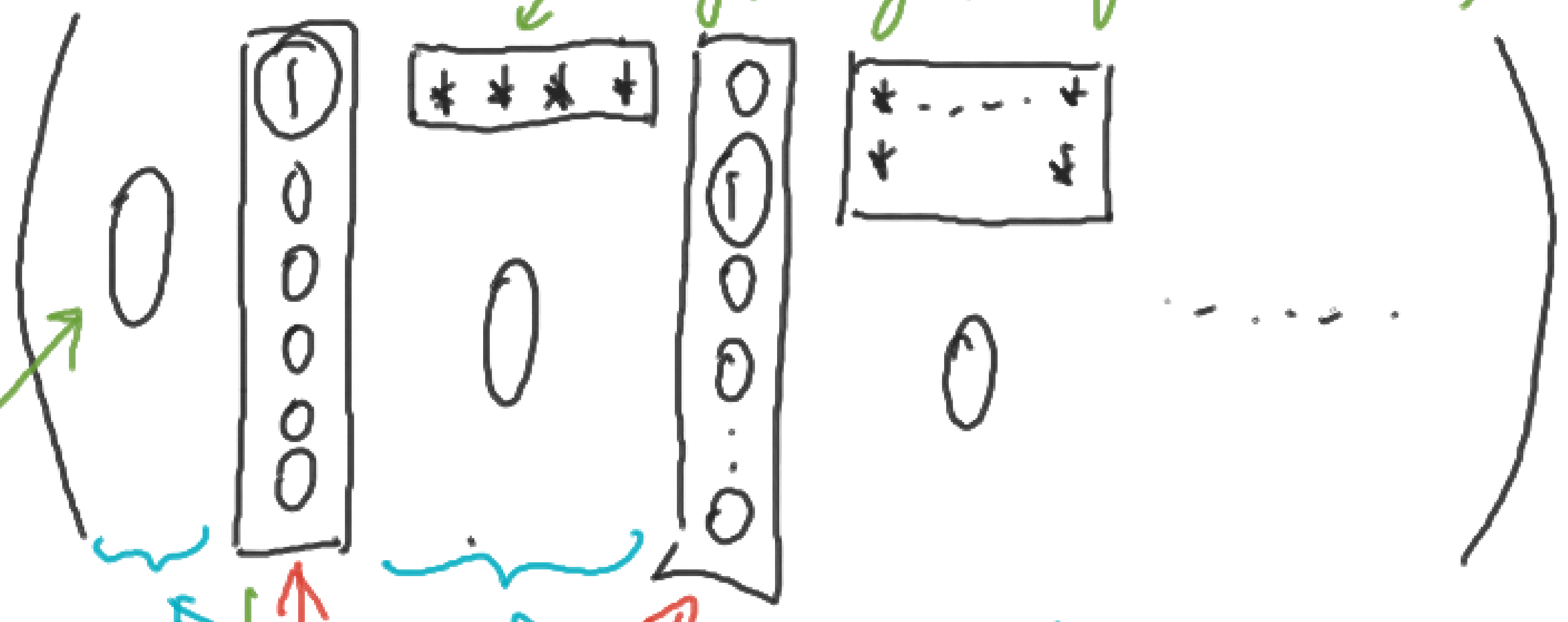
2. Switch two rows

3. Multiply a row by a non-zero number λ $\lambda \neq 0$

But what is our goal (in mathematical terms)?

Reduced row echelon form (RREF)

anything (may not occur)



The RREF applies with or without the vertical line

they may or may not be there

a pivot column. The pivot is in the first row that doesn't have a pivot already. Only 0's above and below pivot allowed

non-pivot columns: any numbers in row of pivots to the left, 0's below

The role of Gauss elimination is using elementary row operations to get to RREF.

Key application: Solving systems of linear equations.

Example:

$$\begin{aligned} x + 2y + z + 4t &= 10 \\ z + t &= 5 \\ 2x + 3y + t &= 6 \\ 3x + 5y + z + 5t &= 16 \end{aligned}$$

Check:

$$(-3+7A)+(8-10A)+(5-A)+4A = 10$$
$$(5-A)+A = 5$$
$$(-6+14A)+(12-15A)+A = 6$$
$$(-9+21A)+(20-25A)+(5-A)+5A = 16$$

✓

$$\begin{aligned} x &= -3+7A \\ y &= 4-5A \\ z &= 5-A \\ t &= A \end{aligned}$$

Strategy: ⇝ RREF

$$\begin{array}{cccc|c} x & y & z & t & \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 10 \\ 0 & 0 & 1 & 1 & 5 \\ 2 & 3 & 0 & 1 & 6 \\ 3 & 5 & 1 & 5 & 16 \end{array}\right) \begin{array}{c} -2 \\ -3 \end{array}$$

(OK continue adding multiples of the same row to other rows)

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 10 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & -1 & -2 & -7 & -14 \\ 0 & -1 & -2 & -7 & -14 \end{array}\right) -1$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 10 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & -1 & -2 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

typically, you would not write the 0 row

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 10 \\ 0 & -1 & -2 & -7 & -14 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) -1$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 10 \\ 0 & 1 & 2 & 7 & 14 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) -2$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & -10 & -18 \\ 0 & 1 & 2 & 7 & 14 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} -2 \\ 3 \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -7 & -3 \\ 0 & 1 & 0 & 5 & 4 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

non-pivot columns

To solve the equations from here:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -3+7A \\ 4-5A \\ 5-A \\ A \end{pmatrix}$$

from the RHS

− non-pivot numbers in the matrix

non-pivot columns to the left of the line give parameters

Example: (without vertical line)

Convert the following matrix to RREF using elementary row operations:

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 5 \\ 2 & 4 & 6 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 \\ 2 & 4 & 6 & 10 \end{pmatrix} \quad -2$$

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad -3$$

non-pivot columns

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns

(HW) (3) Solve using Gauss elimination:

$$\begin{aligned} 2x + z &= 10 \\ x + y + 3z + t &= 5 \\ 2x + 3y + t &= 1 \end{aligned}$$

(4) Convert to RREF using elementary row operations:

$$\begin{pmatrix} 0 & 3 & 6 & 3 & 6 & 0 \\ 0 & 2 & 4 & 3 & 5 & 1 \\ 0 & 1 & 2 & 5 & 6 & 4 \\ 0 & 4 & 8 & 1 & 5 & -3 \end{pmatrix}$$