MATH 417 1/13/2023

To solve systems of linear equations, we write them in matrix form and then use elementary row operations to get the matrix to RREF.

Solving systems of linear equations in RREF

Example: Solve the system in matrix form in the unknowns x, y, z, t, u, v, w, r.

$$\begin{array}{c}
\begin{array}{cccccccc|c} x & y & z & t & u & v & w & r & \end{array} \\
\left(\begin{array}{cccccccc|c}
0 & 1 & -2 & 0 & 3 & 5 & 0 & 1/2 & 3 \\
0 & 0 & 0 & 1 & 4 & 1 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1/3 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \\
\begin{array}{cccccccc} \uparrow & & \uparrow & & \uparrow & \uparrow & & \uparrow \\ A & & B & & C & D & & E \end{array}
\end{array}$$

non-pivot columns $\Rightarrow$ introduce parameters

$$y - 2z + 3u + 5v + \frac{r}{2} = 3$$
$$t + 4u + v + r = 3$$
$$w - \frac{r}{3} = 6$$

Solution:

$$\begin{pmatrix} x \\ y \\ z \\ t \\ u \\ v \\ w \\ r \end{pmatrix} = \begin{pmatrix} A \\ 3 + 2B - 3C - 5D - \frac{E}{2} \\ B \\ 3 - 4C - D - E \\ C \\ D \\ 6 + E/3 \\ E \end{pmatrix}$$

If the matrix of the system of linear equations is in RREF then the non-pivot columns to the left of the line correspond to parameters and the equations in RREF express the other unknowns in terms of the parameters.

The system has no solution if there is a pivot to the right of the vertical line.

Example:
$$\left(\begin{array}{ccccccc|c} 1 & 0 & 2 & 0 & 3 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

This system has no solution

The last equation says $0=1$, which is false. Therefore, no values of the unknowns can satisfy this condition.

Note: Sometimes, we can see this *without* determining the RREF completely.

Example: Does the following system of linear equations in matrix form have a solution?

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 4 \\ 2 & 1 & 3 & -1 & 6 \\ 4 & 1 & 5 & -1 & 9 \end{array}\right)$$

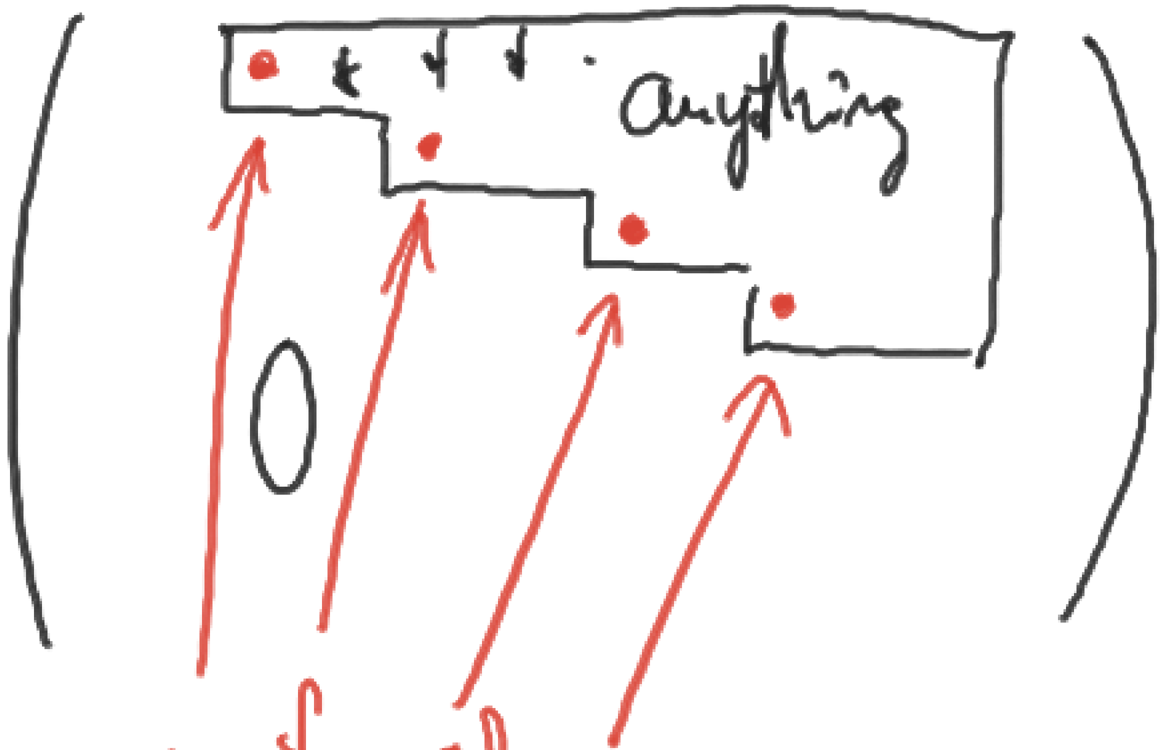
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 4 \\ 2 & 1 & 3 & -1 & 6 \\ 4 & 1 & 5 & -1 & 9 \end{array}\right) \begin{array}{l} \cdot(-2) \\ \cdot(-4) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 4 \\ 0 & -3 & -3 & 3 & -2 \\ 0 & -7 & -7 & 7 & -7 \end{array}\right) \cdot(-7/3)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & -2 & 4 \\ 0 & -3 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 & \underbrace{-7+\frac{14}{3}}_{-\frac{7}{3} \neq 0} \end{array}\right)$$

Answer: $\boxed{\text{NO}}$

A matrix is in Row Echelon Form (REF) if it is of the form



pivots always non-zero and in the next available row

Clearly, a REF is not uniquely determined. For example, on the previous page, the last matrix was in REF, but not in RREF.

Question: Is the RREF uniquely determined by the matrix A (and the requirement that it be obtained from the matrix A by elementary row operations?

Answer: Yes, the RREF is uniquely determined by the initial matrix A.

Why is this not obvious?

Example: Find the RREF of the matrix

$$A = \begin{pmatrix} 2 & 2 & 4 & 6 \\ 3 & 4 & 4 & 2 \end{pmatrix}$$

Solution 1:

$$\begin{pmatrix} 2 & 2 & 4 & 6 \\ 3 & 4 & 4 & 2 \end{pmatrix} \; -1$$

$$\begin{pmatrix} 2 & 2 & 4 & 6 \\ 1 & 2 & 0 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & -4 \\ 2 & 2 & 4 & 6 \end{pmatrix} \; -2$$

$$\begin{pmatrix} 1 & 2 & 0 & -4 \\ 0 & -2 & 4 & 14 \end{pmatrix} \; -1/2$$

$$\begin{pmatrix} 1 & 2 & 0 & -4 \\ 0 & 1 & -2 & -7 \end{pmatrix} \; -2$$

$$\boxed{\begin{pmatrix} 1 & 0 & 4 & 10 \\ 0 & 1 & -2 & -7 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & 2 & 4 & 6 \\ 3 & 4 & 4 & 2 \end{pmatrix} \; 1/2$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 4 & 2 \end{pmatrix} \; -3$$

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -2 & -7 \end{pmatrix} \; -1$$

$$\boxed{\begin{pmatrix} 1 & 0 & 4 & 10 \\ 0 & 1 & -2 & -7 \end{pmatrix}}$$

Same answer. How come?!

HW 5 Solve the following system of linear equations in matrix form

$$\left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

where the unknowns corresponding to the columns (in the order left-to-right) are x, y, z, t, u, v

6 Does the following system have a solution?

$$\left(\begin{array}{cccc|c} 2 & 3 & 5 & 9 & 8 \\ 1 & 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 7 & 5 \end{array}\right).$$

Use REF.