MATH 417 1/18/2023

Why is the RREF of a matrix uniquely determined?

A <u>linear combination</u> is a combination of addition and taking scalar multiples.

<u>Example with column vectors</u>: Compute the linear combination

$$3\begin{pmatrix}1\\2\\4\end{pmatrix} + (-1)\begin{pmatrix}2\\1\\0\end{pmatrix} + 4\begin{pmatrix}1\\0\\1\end{pmatrix}$$

these numbers are called <u>coefficients</u>

<u>Solution</u>: $\begin{pmatrix}3-2+4\\6-1+0\\12+0+4\end{pmatrix} = \boxed{\begin{pmatrix}5\\5\\16\end{pmatrix}}$

Consider a matrix in RREF:

<u>Example:</u>

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

NO YES NO YES NO

non-pivot columns YES

pivot columns NO

coefficients of the linear combinations are the numbers in the matrix if only using linear combination of pivot columns

Which columns are linear combinations of the columns to their left?

<u>Solution:</u> $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ <u>NO</u> (no columns to the left)

$\begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ <u>YES</u> | $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ <u>NO</u> (1 is not a linear comb. of 0s) | $\begin{pmatrix} 2 \\ -4 \\ 0 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-4) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ <u>yes</u> | $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ <u>NO</u> (all columns to the left had 0s in third row.)

One column being a linear combination of others does not change under row operations

<u>Example</u>: In the matrix $\begin{pmatrix} 1 & 1 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 2 & 4 & 6 \end{pmatrix}$, find out which columns cannot be expressed as linear combinations of columns to their left, and express the remaining ones as linear combinations of those columns.

<u>Solution</u>: Go to RREF.

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 2 & 4 & 6 \end{pmatrix} \; -2 \;\; -1$$

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \; -1$$

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \; -1 \;\; -1$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \end{pmatrix} \; -1$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \; -2 \;\; -1$$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
$$\begin{array}{cccc} \text{NO} & \text{NO} & \text{NO} & \text{YES} \end{array}$$

$$\underset{4}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} = 1 \underset{2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} + 1 \underset{3}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

<u>Answer</u>: 1: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, 2: $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, 3: $\begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$

<u>NO</u>

4: $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}$

<u>YES</u>

Example: Determine which columns of the following matrix cannot be expressed as linear combination of columns to their left and express the other columns as linear combination of those columns:

$$\begin{pmatrix} 1 & 2 & 2 & 4 & 5 \\ 3 & 6 & 5 & 9 & 13 \\ 1 & 2 & 1 & 3 & 3 \end{pmatrix}$$

Solution: Go to RREF:

$$\begin{pmatrix} 1 & 2 & 2 & 4 & 5 \\ 3 & 6 & 5 & 9 & 13 \\ 1 & 2 & 1 & 3 & 3 \end{pmatrix} \quad -3,\ -1$$

$$\begin{pmatrix} 1 & 2 & 2 & 4 & 5 \\ 0 & 0 & -1 & -3 & -2 \\ 0 & 0 & -1 & -1 & -2 \end{pmatrix} \quad \cdot(-1)$$

$$\begin{pmatrix} 1 & 2 & 2 & 4 & 5 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 & -2 \end{pmatrix} \quad -2,\ 1$$

$$\begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix} \quad \cdot\tfrac{1}{2}$$

$$\begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad -3,\ 2$$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$\begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

1, 3, 4 pivots.

2 = 2·1, 5 = 1·1 + 2·3

Answer: 1st, 3rd, 4th columns NO $\begin{pmatrix}1\\3\\1\end{pmatrix}, \begin{pmatrix}2\\5\\1\end{pmatrix}, \begin{pmatrix}4\\9\\3\end{pmatrix}$

$$\begin{pmatrix}2\\6\\2\end{pmatrix} = 2\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}, \quad \begin{pmatrix}5\\13\\3\end{pmatrix} = 1\cdot\begin{pmatrix}1\\3\\1\end{pmatrix} + 2\cdot\begin{pmatrix}2\\5\\1\end{pmatrix}$$

__Observe__: The RREF is, in turn, determined by these linear combination relation between the columns. Hence, it is uniquely determined.

Two matrices are called __row-equivalent__ if you can get from one to the other by elementary row operations. (This happens if and only if they have the same RREF.)

Example: Are the matrices

$$\begin{pmatrix} 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 6 & 2 & 6 \end{pmatrix}$$

row-equivalent?

$$\begin{pmatrix} 2 & 4 & 1 & 3 \\ 1 & 2 & 1 & 2 \end{pmatrix} \qquad\qquad \begin{pmatrix} 1 & 2 & 1 & 2 \\ 3 & 6 & 2 & 6 \end{pmatrix} \; -3$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} \; -2 \qquad\qquad \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \; \cdot(-1)$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix} \; \cdot(-1) \qquad\qquad \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; -1$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \; -1 \qquad\qquad \underbrace{\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\text{RREF}}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}}_{\text{RREF}}$$

different!

$\boxed{\text{ANSWER: NO}}$

HW: (1) In the following matrix, which columns cannot be expressed as linear combinations of the columns to their left? Express the other columns as linear combinations of those columns:

$$\begin{pmatrix} 1 & 3 & 2 & 6 & 7 \\ 1 & 3 & 1 & 5 & 5 \\ 2 & 6 & 1 & 9 & 8 \\ 1 & 3 & 5 & 9 & 13 \end{pmatrix}$$

(2) Are the following matrices row-equivalent? (Justify your answer.)

$$\begin{pmatrix} 2 & 1 & 3 & 5 & 6 \\ 1 & 1 & 2 & 4 & 4 \\ 1 & 2 & 4 & 8 & 7 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 & 4 & 4 \\ 1 & 1 & 2 & 4 & 4 \\ 3 & 2 & 4 & 8 & 9 \end{pmatrix}.$$