MATH 417 1/20/2023

<u>Note</u>: row operations are the same thing as multiplying by some matrix from the left.

We are forming linear combinations of elements of each column of the matrix on the right with the same <u>coefficients</u> (coefficient = entries of the rows of the matrix on the left)

So instead, we could think of it as forming linear combinations of entire rows of the matrix on the right (with those coefficients).

How do we find the matrix corresponding to an elementary row operation? Just do the same operation on the identity matrix.

Example: Find the matrix of the following row operation:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad 4$$

4 × 5 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 & 1 & 5 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 & 2 \\ 1 & 3 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 & 1 & 5 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & -1 & 2 \\ 9 & 7 & 7 & 10 & 5 \end{pmatrix}$$

__Solution:__

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 4$$

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \end{pmatrix}}$$

__Answer__

Why do row operations preserve "linear relationships" between columns

a linear combination being 0

linear dependencies.

$$\underbrace{\begin{pmatrix}\text{row} \\ \text{operation}\end{pmatrix}\begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}\begin{pmatrix}\text{column} \\ \text{operation}\end{pmatrix}}_{\text{Associativity of matrix multiplication.}}$$

<u>Inverse matrix</u>. The inverse matrix of a matrix $A$, if one exists, is a matrix $A^{-1}$ such that $A^{-1}A = AA^{-1} = I$

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Finding the inverse (or determining that one doesn't exist) of a square matrix $A$.

$$AX = I \qquad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

for each column, we have a system of linear equations:

second column (can replace 2 by any $j=1,\dots,n$)

$$\left.\begin{aligned} a_{11}x_{12} + a_{12}x_{22} + \cdots + a_{1n}x_{n2} &= 0 \\ a_{21}x_{12} + a_{22}x_{22} + \cdots + a_{2n}x_{n2} &= 1 \\ \cdots &= 0 \\ a_{n1}x_{12} + a_{n2}x_{22} + \cdots + a_{nn}x_{n2} &= 0 \end{aligned}\right\} \qquad \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & & a_{2n} & 1 \\ \vdots & & & \vdots & 0 \\ & & & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} & 0 \end{array}\right)$$

Combining these systems:

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & & & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right)$$

RREF. Solution when no pivot is to the right of the vertical line

$$\left(\begin{array}{c|c} A & I \end{array}\right)$$

$\Downarrow$ ← elementary row ops.

$$\left(\begin{array}{c|c} I & A^{-1} \end{array}\right)$$

## Example: Find the inverse, if any, of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 3 & 6 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

**Solution:**

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 4 & 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & 6 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & -2 & 1 & 0 & 0 \\ 0 & -2 & -3 & -5 & -2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & -1 & 0 & 0 \\ 0 & -2 & -3 & -5 & -2 & 0 & 1 & 0 \\ 0 & 0 & -3 & -4 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -3 & -4 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & -3 & -4 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -7 & 6 & -3 & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 4 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 7 & -6 & 3 & -1 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -6 & 5 & -2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 3 & -2 & 1 \\ 0 & 0 & 1 & 0 & -9 & 8 & -4 & 1 \\ 0 & 0 & 0 & 1 & 7 & -6 & 3 & -1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} -6 & 5 & -2 & 1 \\ -3 & 3 & -2 & 1 \\ -9 & 8 & -4 & 1 \\ 7 & -6 & 3 & -1 \end{pmatrix}}$$

<u>Example</u>: Find the inverse of the matrix $A$ or determine that none exists where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 8 & 11 \\ 2 & 1 & 3 \end{pmatrix}$$

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<u>Solution</u>:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 3 & 8 & 11 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \quad \cdot 3, \; \cdot 2$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -5 & -5 & -2 & 0 & 1 \end{array}\right) \quad -5$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 & -5 & 1 \end{array}\right)$$

REF

pivot to the right of the line

<u>No</u> solution.

HW ③ Find the matrices of the following row operations:

(a)

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \; -4$$

(b)

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

(c)

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \; 1/2$$

④ Find the inverse of the matrix A or determine that none exists:

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

⑤ Find the inverse of the matrix A or determine that none exists:

$$A = \begin{pmatrix} -1 & 0 & -1 & -2 \\ 1 & 1 & 2 & 4 \\ 2 & 3 & 4 & 9 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

due Monday 1/23
10AM