MATH 417 1/23/2023

Last time: let A be a square matrix. Does there exist a matrix A^{-1} such that

$$AA^{-1} = I \qquad (1)$$

$$A^{-1}A = I \qquad (2)$$

Not in general. But (1) can be solved as a system of linear equations: If

$$(A|I) \sim (I|B)$$

note: B uniquely determined

elementary row operations

$$(A|I) \sim (RREF)$$

with proof to right side of vertical line

then there is no solution.

Then we have solved (1): $AB = I$. What about (2)?

Suppose we found (A square)

$$(A|I) \sim (I|B)$$

Then we know that $AB = I$. Is it necessarily true that $BA = I$?

Answer: Yes. But why?   multiply by A on the right

If $AB = I$ then $ABA = A$. Assuming there is a C such that $CA = I$, then we can conclude

$$BA = CABA = CA = I.$$

Do we know that there is a C such that $CA = I$?

Yes, we know the existence of C with $CA = I$ from

$$(A|I) \sim (I|B).$$

We can convert $(A|I)$ to $(I|B)$ by elementary row operations. But elementary row operations are the same thing as multiplying on the left by some matrices $M_1, M_2, \dots, M_k$.

$$M_k M_{k-1} \cdots M_2 M_1 (A|I) = (I|B)$$

In particular,

$$M_k M_{k-1} \cdots M_2 M_1 A = I$$

So we can just put $C = M_k M_{k-1} \cdots M_2 M_1$.

But note: We do not have to do this in practice. We already proved that then $BA = I$.

But can we conclude that $C = B$? If A is a square matrix, is a matrix B such that $BA = I$ uniquely determined? (Provided it exists.)

We already showed that the solution B to $AB = I$ is uniquely determined (because $(A|I) \sim (I|B)$ means no free parameters)

We have matrix transposition (works for any matrix, not necessarily square):

"A transposed"

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

$m \times n$ matrix.

Example: Find A^T where

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & -4 \end{pmatrix}$$

Solution:

$$A^T = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 1 & -4 \end{pmatrix}.$$

We have $(AB)^T = B^T A^T$

$$\underbrace{\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & -4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 25 \\ -7 \end{pmatrix}}_{AB} \qquad \underbrace{\begin{pmatrix} 1 & 7 & 2 \end{pmatrix}}_{B^T} \underbrace{\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 1 & -4 \end{pmatrix}}_{A^T} = \underbrace{(25, -7)}_{(AB)^T}.$$

"every statement that works for matrix multiplication from one side also works, by transposition, works from the other side."

"If A is a square matrix then a solution B to $AB = I$ (if any) is uniquely determined."

This also means that a solution C to $CA = I$ (if any) is uniquely determined.

$CA = I$ happens if and only if $A^T C^T = I \; (= I^T)$

But then C^T is uniquely determined. Therefore, C is uniquely determined.

To recapitulate: To solve $A^{-1}A = AA^{-1} = I$.

$(A|I) \sim (\text{RREF})$ If the RREF is $(I|B)$ then $A^{-1} = B$ is the uniquely determined inverse of A. Otherwise, A has no inverse. This works for a <u>square matrix</u> A.

<u>Example</u>: Find an inverse, if any, of $A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 3 & 5 \end{pmatrix}$

<u>Solution</u>:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 \\ 4 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ 0 & -1 & 1 & 0 & -4 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -7 & 3 \\ 0 & 1 & 0 & 1 & 6 & -2 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} -2 & -7 & 3 \\ 1 & 6 & -2 \\ 1 & 2 & -1 \end{pmatrix}}$$

HW

1. Find the inverse of the matrix A or conclude that none exists:

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 1 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}$$

2. Find the inverse to the matrix A or conclude that none exists:

$$A = \begin{pmatrix} 2 & 1 & 3 & 6 \\ 1 & 1 & 2 & 4 \\ 2 & 4 & 5 & 11 \\ 3 & 1 & 1 & 5 \end{pmatrix}$$