MATH 417 1/25/2023

What about inverse when matrices are not square?

We could attempt to solve $AX = I$.

<u>Example</u>: Find a matrix X such that

$$\begin{pmatrix} 2 & 4 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

if possible.

<u>Solution</u>:

$$\left(\begin{array}{cccc|cc} 2 & 4 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 3 & 0 & 1 \end{array}\right) \updownarrow$$

$$\left(\begin{array}{cccc|cc} 1 & 2 & 1 & 3 & 0 & 1 \\ 2 & 4 & 1 & 1 & 1 & 0 \end{array}\right) -2$$

$$\left(\begin{array}{cccc|cc} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & -1 & -5 & 1 & -2 \end{array}\right) -1$$

$$\left(\begin{array}{cccc|cc} 1 & 2 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 5 & -1 & 2 \end{array}\right) -1$$

$$\left(\begin{array}{cccc|cc} 1 & 2 & 0 & -2 & 1 & -1 \\ 0 & 0 & 1 & 5 & -1 & 2 \end{array}\right)$$

Answer should be a 4×2 matrix

$$\begin{pmatrix} 1-2a+2b & -1-2c+2d \\ a & c \\ -1-5b & 2-5d \\ b & d \end{pmatrix}$$

e.g. $a=b=c=d=0$

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 5 & -1 \end{array}\right) \\ \begin{array}{cccc} & \uparrow & & \uparrow \\ & a & & b \end{array} \end{array}$$

$$\begin{array}{c} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -2 & -1 \\ 0 & 0 & 1 & 5 & 2 \end{array}\right) \\ \begin{array}{cccc} & \uparrow & & \uparrow \\ & c & & d \end{array} \end{array}$$

⊛ <u>Example</u>: Solve

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

<u>Solution</u>:

$$\left(\begin{array}{cc|cccc} 2 & 1 & 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

This cannot work: We are not allowed any pivots to the right of the line. So we are allowed at most two pivots $\Rightarrow$ at least two 0 lines in RREF.

But the unit matrix is not row-equivalent to a matrix with a 0 row (because it is in RREF and the RREF is uniquely determined).

In general, #

$$m\left\{\underbrace{\begin{pmatrix} \; A \; \end{pmatrix}}_{n}\right. X = \left.\begin{pmatrix} \quad I \quad \end{pmatrix}\right\} m$$

<u>cannot</u> be solved when $m > n$.

This is because

$$\left(\begin{array}{c|c} A & I \end{array} \right)$$

is not allowed to have a pivot to the right of the line if there is to be a solution. In order for no pivot to occur to the right of the line, there would be a 0 row in RREF. But I is not row equivalent to a matrix with a 0 row (because RREF is unique).

Note: Size does not guarantee a solution

Example: Find a matrix X (if any) such that

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 2 & 0 & 2 & 2 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{cccc|ccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 3 & 5 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -1 \\ -2 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 1 & 0 \\ 0 & -2 & -2 & -4 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} 2 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 & 2 & 1 \end{array}\right)$$

REF $\therefore$ No solution.

__Example:__ Solve (if possible)

$$Z\begin{pmatrix} 2 & 4 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1)$$

__Solution:__ Refer to example 4. Setting $Y = Z^T$, equation (1) implies

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} Z^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which has no solution because of size.

$$Z \left.\underbrace{\begin{pmatrix} \; A \; \end{pmatrix}}_{n}\right\} m = \left.\underbrace{\begin{pmatrix} \;\; I \;\; \end{pmatrix}}_{n}\right\} n$$

is not solvable if $m < n$.

(Perform a transposition and refer to page #.)

A matrix X such that $AX = I$ is called a <u>right inverse</u> to A.

A matrix Y such that $YA = I$ is called a <u>left inverse</u> to A.

A right (or left) inverse <u>does not exist</u> when I has more entries than the matrix A.

Otherwise, the left or right inverse may or may not exist.
Use Gauss elimination. (for finding left inverse, transpose, and if you find a solution, transpose it back)

An inverse from both sides can only exist when A is square. In that case, inverses from both sides (if an inverse from one side exists) are equal and uniquely determined.

HW 3 Determine whether the matrix

$$A = \begin{pmatrix} 3 & 5 & 8 & 11 & 16 \\ 1 & 2 & 3 & 4 & 6 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}$$

a) Has a left inverse. If so, find all its left inverses

b) has a right inverse. If so, find all its right inverses.

(4) Determine if the matrix

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{pmatrix}$$

(a) Has a left inverse. If so, find all its left inverses.

(b) Has a right inverse. If so, find all its right inverses.