MATH 417 1/27/2023

Example: Find a left inverse, if one exists, of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

Find a right inverse to A^T.

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} -2 \\ -3 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} -1 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 2 & -1 & 0 \\ 0 & -1 & 0 & -1 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} -2 \\ 1 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 3 & -1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 1 \end{array}\right) \begin{array}{l} -1 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 3 & -1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{array}\right) \begin{array}{l} -3 \\ 1 \end{array}$$

$$\left(\begin{array}{cccc|ccc} 1 & 0 & 0 & -1 & -6 & -1 & 3 \\ 0 & 1 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{array}\right)$$

general solution to

$$\begin{pmatrix} -6+a & -1+b & 3+c \\ 3-a & -b & -1-c \\ 1 & 1 & -1 \\ a & b & c \end{pmatrix}$$

Answer:

$$\begin{pmatrix} -6+a & 3-a & 1 & a \\ -1+b & -b & 1 & b \\ 3+c & -1-c & -1 & c \end{pmatrix}$$

<u>Remark</u>: The <u>rank</u> of a matrix is the number of pivots in its RREF.

<u>Fact</u>: $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

Having an inverse for an $n \times n$ matrix is equivalent to having rank n.

Having a <u>right inverse</u> for an $m \times n$-matrix A is equivalent to $\operatorname{rank} A = m$. (number of rows)

Having a <u>left inverse</u> for an $m \times n$-matrix A is equivalent to $\operatorname{rank} A = n$. (number of columns)

Example: Find rank(A) where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \\ 4 & 1 & 5 \\ 5 & 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \\ 4 & 1 & 5 \\ 5 & 1 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \\ 4 & 1 & 5 \\ 5 & 1 & 6 \end{pmatrix} \quad \cdot(-2),\ -3,\ -4,\ -5$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & -3 & -3 \\ 0 & -4 & -4 \end{pmatrix} \quad -3,\ -4$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

REF

good enough for finding rank

$\operatorname{rank}(A) = 2$

Another method:

$$A^T = \begin{pmatrix} 2 & 1 & 3 & 4 & 5 \\ 1 & 1 & 3 & 1 & 1 \\ 3 & 2 & 6 & 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 3 & 1 & 1 \\ 2 & 1 & 3 & 4 & 5 \\ 3 & 2 & 6 & 5 & 6 \end{pmatrix} \begin{matrix} \\ -2 \\ -3 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 3 & 1 & 1 \\ 0 & -1 & -3 & 2 & 3 \\ 0 & -1 & -3 & 2 & 3 \end{pmatrix} \begin{matrix} \\ \\ -1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 3 & 1 & 1 \\ 0 & -1 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{REF}$$

$$\text{rank}(A^T) = 2 = \text{rank}(A)$$

I owe you the proof of $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

The geometric meaning of a matrix is a linear transformation

$\mathbb{R}^m$ is the set of all column vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$.

Set-theoretical notation:

$$\mathbb{R}^m = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \middle| \, x_1, \dots, x_m \in \mathbb{R} \right\} = \left\{ (x_1, \dots, x_m)^T \,\middle|\, x_1, \dots x_n \in \mathbb{R} \right\}.$$

Given a matrix $\underset{m \times n}{A}$, the **linear transformation** corresponding to A assigns to a vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ the vector $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Linear transformations are a special kind of **mappings**:
A mapping $f: S \longrightarrow T$ where $\overset{\text{domain}}{S}$ and $\overset{\text{codomain}}{T}$ are sets assigns to every element $x \in S$ precisely one element $f(x) \in T$.

Example: Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation corresponding to the matrix

$$\begin{pmatrix} 2 & 1 & 4 \\ 1 & -1 & 10 \end{pmatrix}.$$

Find the formula for $f\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ without using matrices.

Solution:

linear expressions, NO CONSTANT TERM

answer

$$\begin{pmatrix} 2 & 1 & 4 \\ 1 & -1 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boxed{\begin{pmatrix} 2x + y + 4z \\ x - y + 10z \end{pmatrix} = f\begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

HW 5 Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 1 & 5 & 6 \\ 0 & 0 & 1 & 1 & 2 \\ 3 & 6 & 2 & 8 & 10 \end{pmatrix}$$

6 Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear transformation corresponding to the matrix

$$\begin{pmatrix} 2 & 6 & 2 \\ 1 & 5 & 0 \\ -10 & 0 & 1 \\ 3 & 4 & 1 \end{pmatrix}$$

Find a formula for $f\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that does not use matrices.