MATH 417 1/30/2023

Proof of the fact that

$$\operatorname{rank} A = \operatorname{rank} A^T.$$

defined as number of pivots in RREF

Instead of RREF of A^T, think of Reduced Column Echelon form of A itself, attained by doing elementary column operations.
We know that every column of A is a linear combination of pivot columns. So I can eliminate every non-pivot column doing elementary column operations. So I have only pivot columns left. So there will be at most that many pivots in RCEF.

Transposing, we can conclude that

$$\operatorname{rank} A^T \leq \operatorname{rank} A$$

By replacing A by A^T, we also get

$$\operatorname{rank} A \leq \operatorname{rank} A^T.$$

Therefore,

$$\operatorname{rank} A = \operatorname{rank} A^T.$$

Back to linear transformations. $\mathbb{R}^n$ is the set of all column vectors of length n ($n \times 1$ matrices). An $m \times n$ matrix A defines a mapping $f: \mathbb{R}^n \to \mathbb{R}^m$

$$x \longmapsto Ax.$$

This is called a <u>linear transformation</u>

linear mapping $\parallel$

<u>Example</u>: Find a formula (without using matrices) for the linear mapping $\mathbb{R}^2 \to \mathbb{R}^2$ corresponding to the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

<u>Solution</u>:

$$\begin{pmatrix} z \\ t \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix} = \begin{pmatrix} 2z+t \\ z+3t \end{pmatrix}$$

Entries are linear combinations of the independent variables z, t.

Going backwards: If I am given a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ by an explicit formula, when is it linear and how do I find its matrix?

Defining properties of a linear mapping (= transformation) $f : \mathbb{R}^n \to \mathbb{R}^m$:

$$\boxed{\begin{array}{ll} f(x+y) = f(x) + f(y) & x, y \in \mathbb{R}^n \\ f(ax) = a f(x) & a \in \mathbb{R} \end{array}}$$

Second property implies $f(0) = 0$.

In "layman's terms": Each coordinate of the dependent vector is a linear combination of the coordinates of the independent vector.

Example: (a) Is the mapping $f: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y \\ x + 10 \\ x + \sin y \end{pmatrix}$$

linear? If so, find its matrix.

(b) Same question for the mapping $g: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x + 4y \\ 3y \end{pmatrix}$$

Solution: (a) The expressions are not linear combinations (all three fail, but it would be enough to fail to conclude that): IT IS NOT LINEAR

(b) $$\begin{pmatrix} 2x + 3y \\ x + 4y \\ 3y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Answer: $\begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 0 & 3 \end{pmatrix}$

Notice:

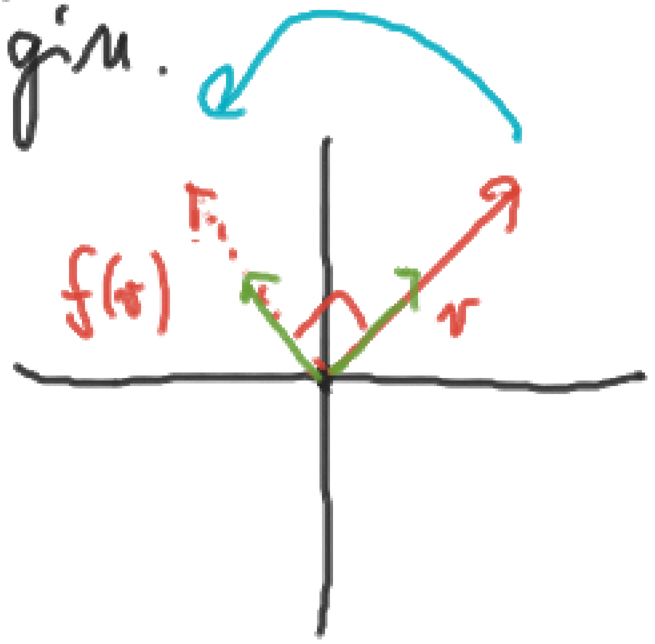
$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

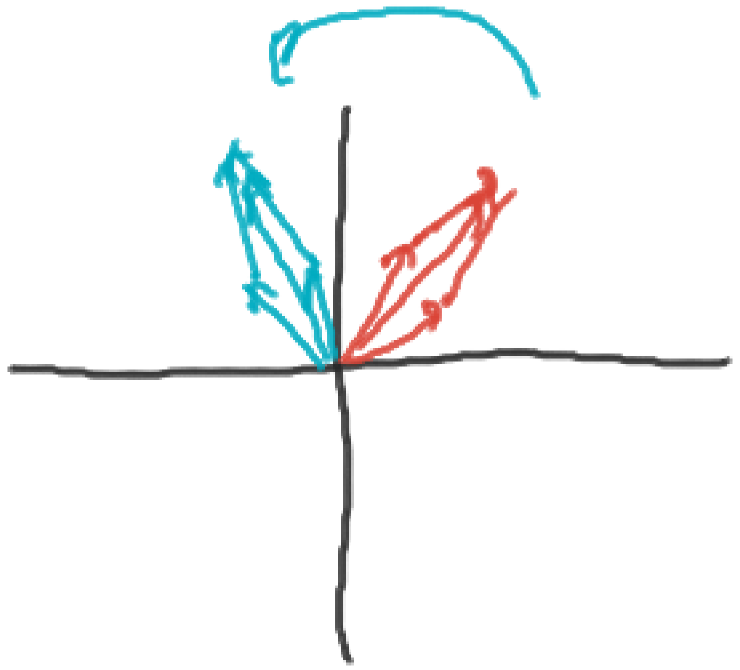
$$\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} = g\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To find the matrix of a linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, first check (at least mentally) that it really is linear (i.e. that it satisfies the defining properties ⊛) and then the matrix of f is

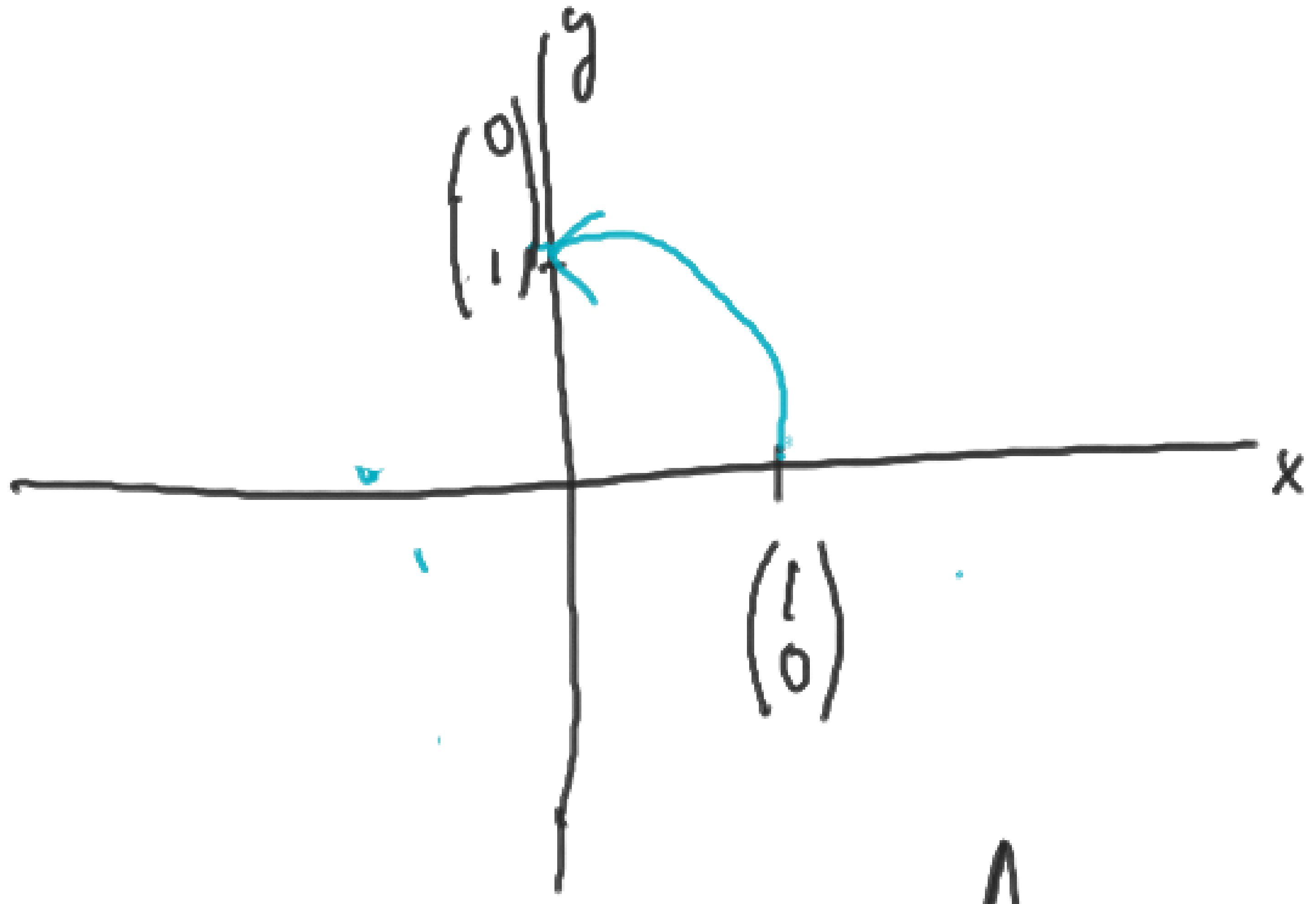
$$\left(f\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} \quad f\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \quad \cdots \quad f\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix} \right).$$

<u>Example</u>: In $\mathbb{R}^2$, find the matrix of the rotation 90° counterclockwise about the origin.



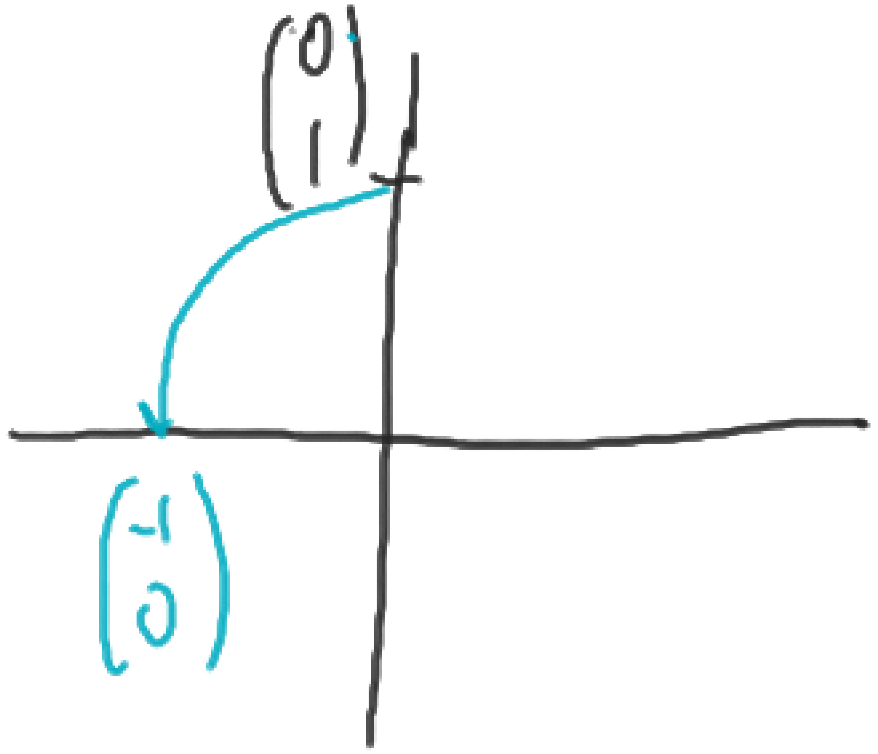


we verify that this mapping is linear



$$\begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\end{pmatrix}$$

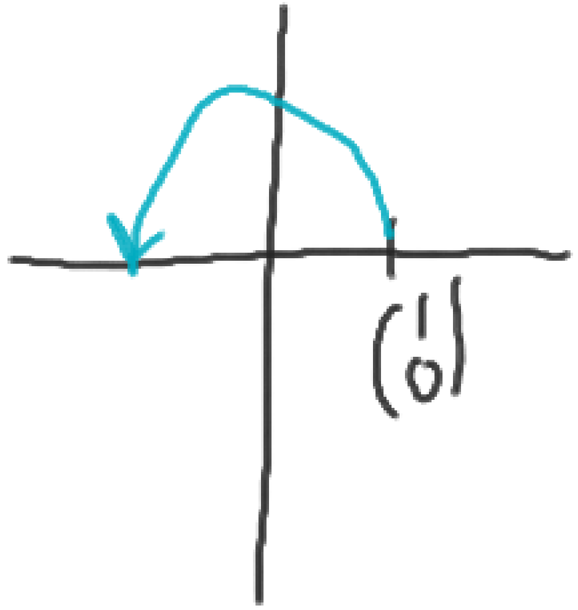
$$\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}-1\\0\end{pmatrix}$$



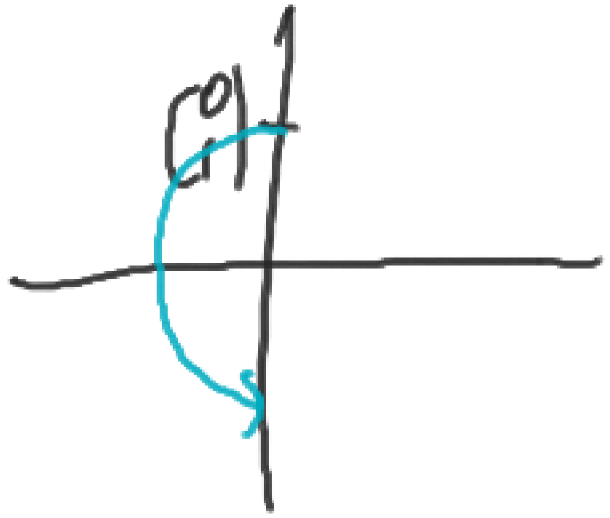
Answer: $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

Example: Find the matrix of the rotation $f: \mathbb{R}^2 \to \mathbb{R}^2$ 180° by the origin.

Solution:



$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

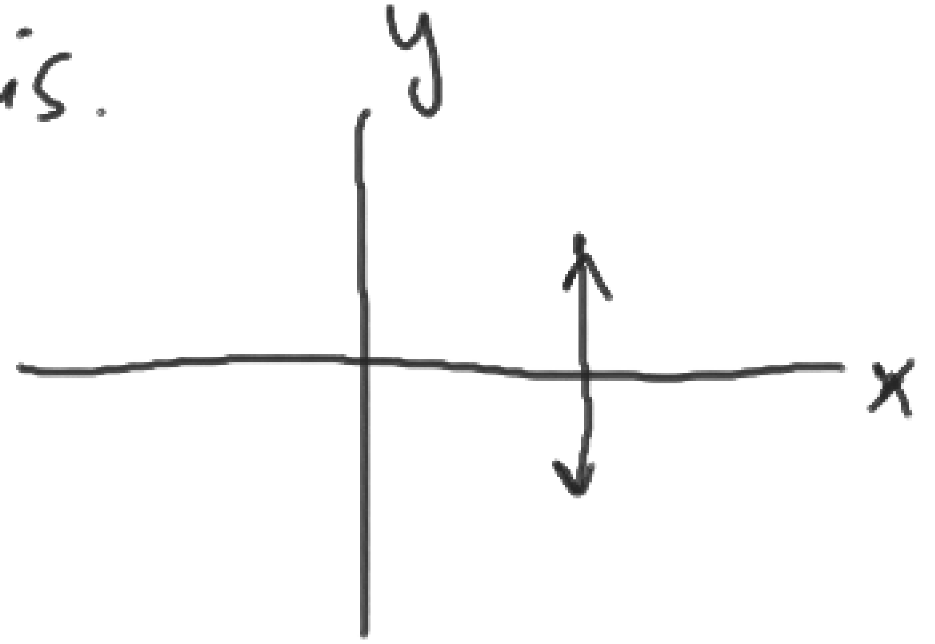


$$f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

Answer:

$$\boxed{\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}}$$

HW: 1 Find the matrix of the reflection in $\mathbb{R}^2$ by the x axis.



2 Find the matrix of the rotation in $\mathbb{R}^2$ 90° clockwise about the origin.