MATH 417 2/1/2023

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then its matrix A is

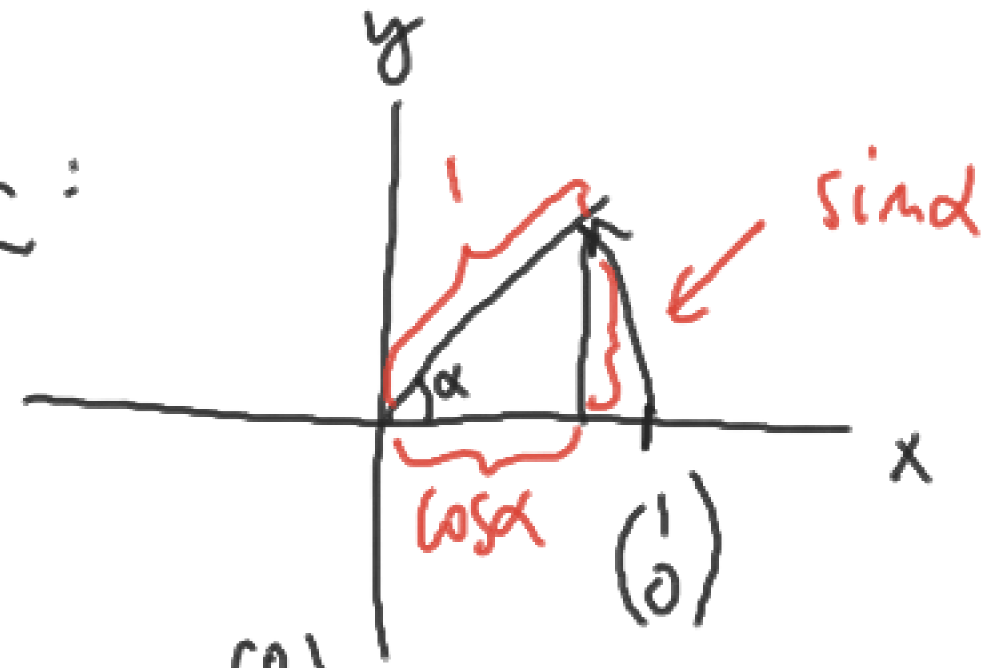
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$A = \left(f\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots f\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right)$$

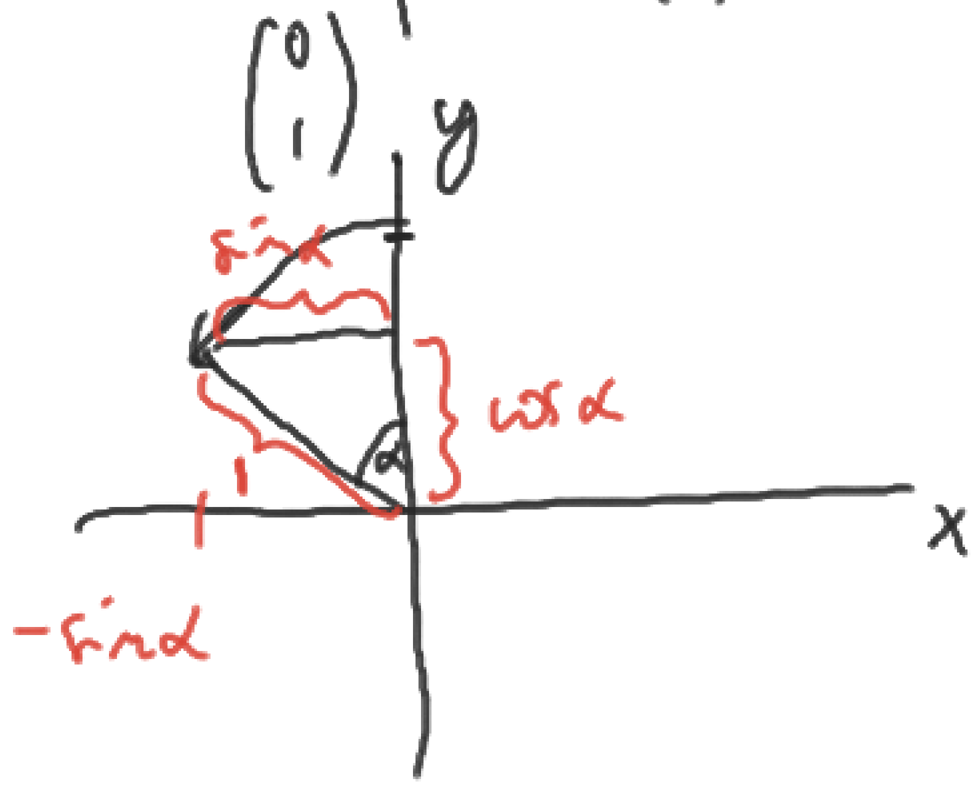
$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 f\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n f\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \underbrace{\left(f\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots f\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)}_{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Example: Find the matrix of the rotation by an angle α counterclockwise around the origin in $\mathbb{R}^2$.

Solution:



$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}$$



$$f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}$$

$$\boxed{\text{Answer: } \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}}$$

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

<u>Example</u>: Find the point given by rotating the point $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ 45° counterclockwise about the origin.

<u>Solution</u>: $\cos 45^\circ = \frac{1}{\sqrt{2}} = \sin 45^\circ$

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{pmatrix}}}$$

Composition of mappings: $f : S \to T$, $g : T \to U$

$$S \xrightarrow{f} T, \quad T \xrightarrow{g} U$$

$g \circ f : S \to U$ $\qquad g \circ f(x) = g(f(x))$

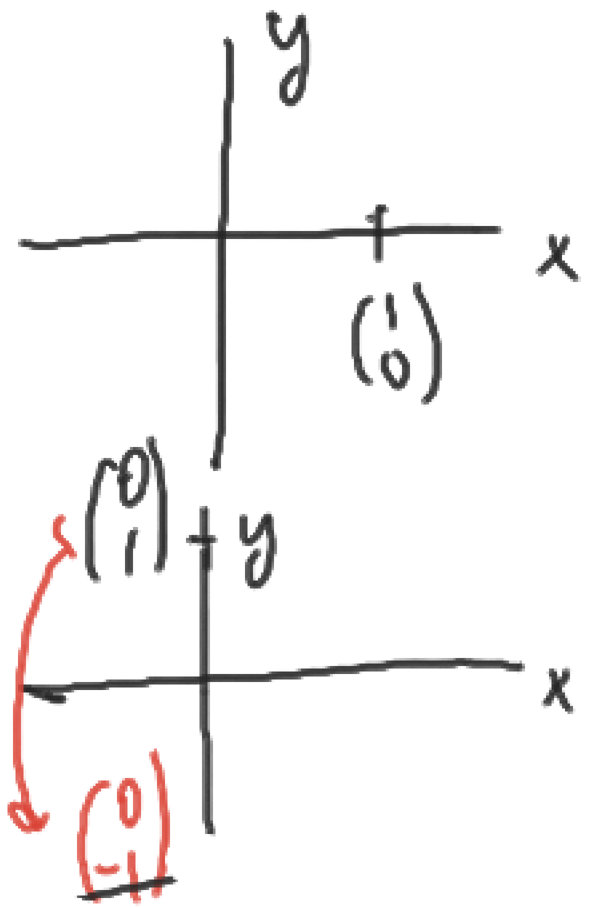
$$\begin{tikzcd} S \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & T \arrow[r, "g"] & U \end{tikzcd}$$

If $\underbrace{f : \mathbb{R}^n \to \mathbb{R}^m}_{\text{has matrix } A}$, $\underbrace{g : \mathbb{R}^m \to \mathbb{R}^k}_{\text{has matrix } B}$ are linear transformations

then $g \circ f$ has matrix BA. $\qquad g \circ f(x) = g(f(x)) = g(Ax) = BAx$

Example: Find the matrix of the reflection in $\mathbb{R}^2$ by the x-axis.

Solution:



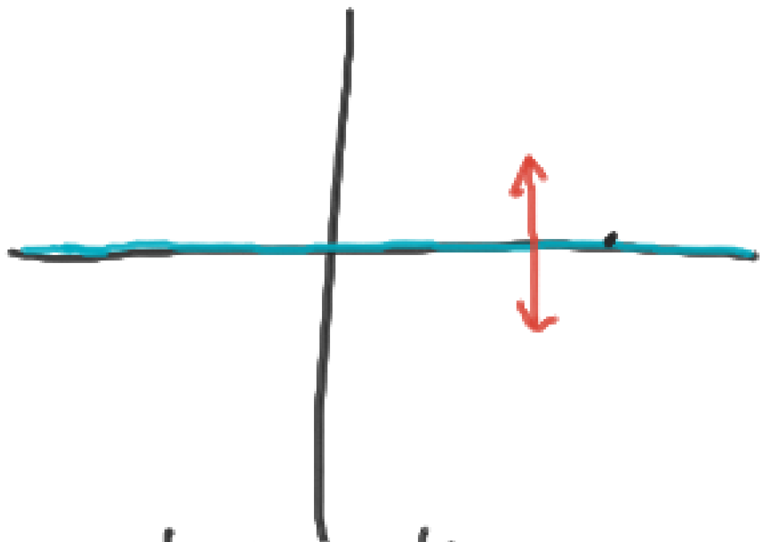
$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

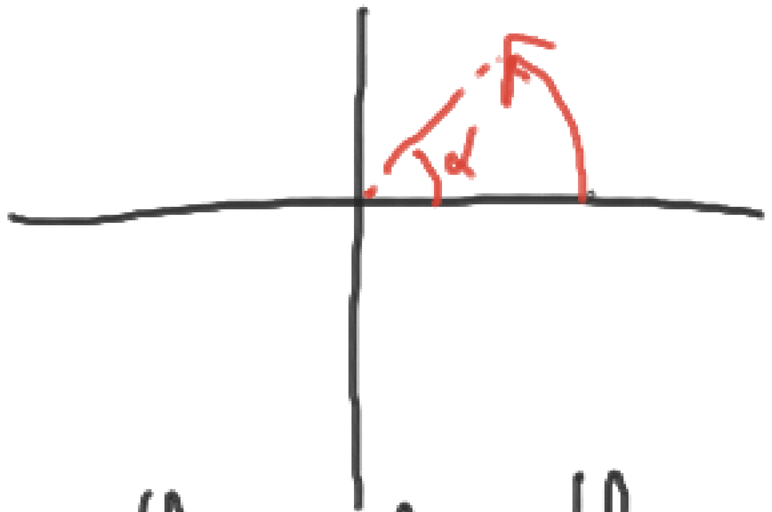
Answer: $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

Matrix of reflection by the x axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

What happens if I compose the reflection about the x axis (done first) with a rotation by the angle α counterclockwise about the origin?



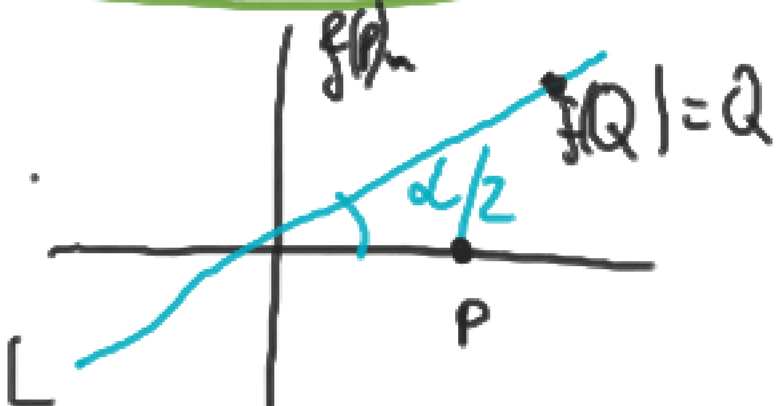
first do this



then do this

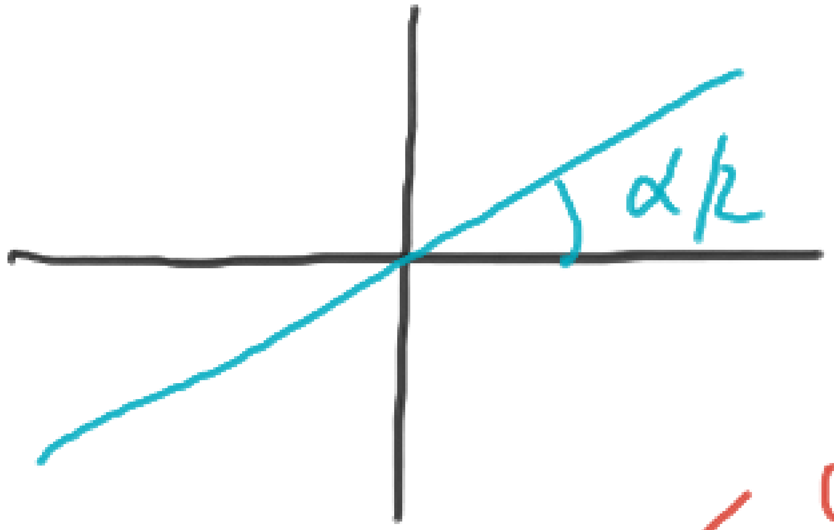
true for points on the x axis and for point on the line L.

reflection about this line.



the axis which is the x line rotated by the angle $\alpha/2$ counterclockwise.

The matrix of the reflection by the line L:



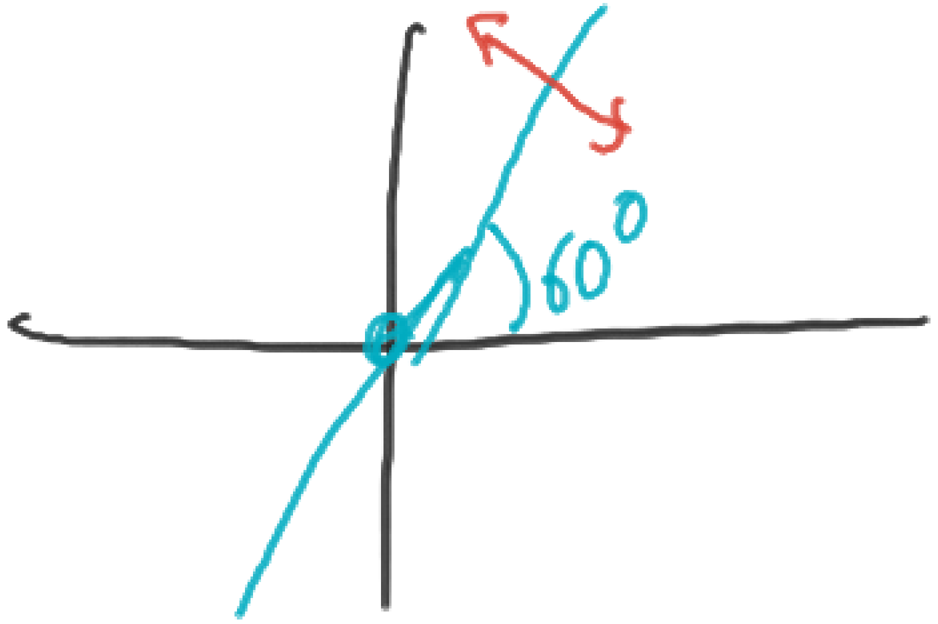
rotation by the angle α counterclockwise

reflection by the x axis

$$\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha\end{pmatrix}$$

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

<u>Example</u>: Find the matrix of the reflection in $\mathbb{R}^2$ about the line obtained by rotating the x axis 60° counterclockwise around the origin.



<u>Solution</u>: $60^\circ = \alpha/2 \qquad \alpha = 120^\circ.$

$$\cos 120^\circ = -1/2$$

$$\sin 120^\circ = \sqrt{3}/2$$

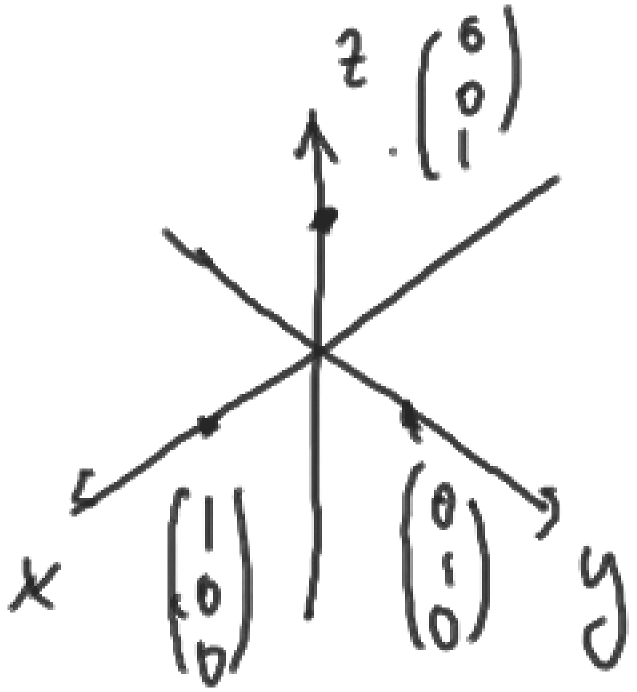
$$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

Example: Find the matrix of the following transformation in $\mathbb{R}^3$:

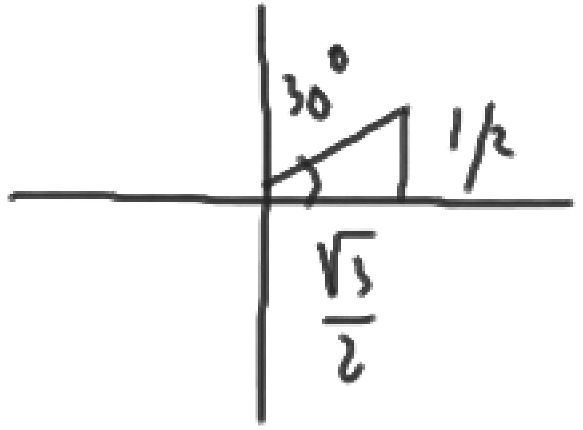
First perform a rotation in the xy plane 30° from x to y.

the positive part

Then perform a rotation in the xz plane 45° from (positive part of) the x axis to (the positive part of) the z axis.



30° degrees from x to y



$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

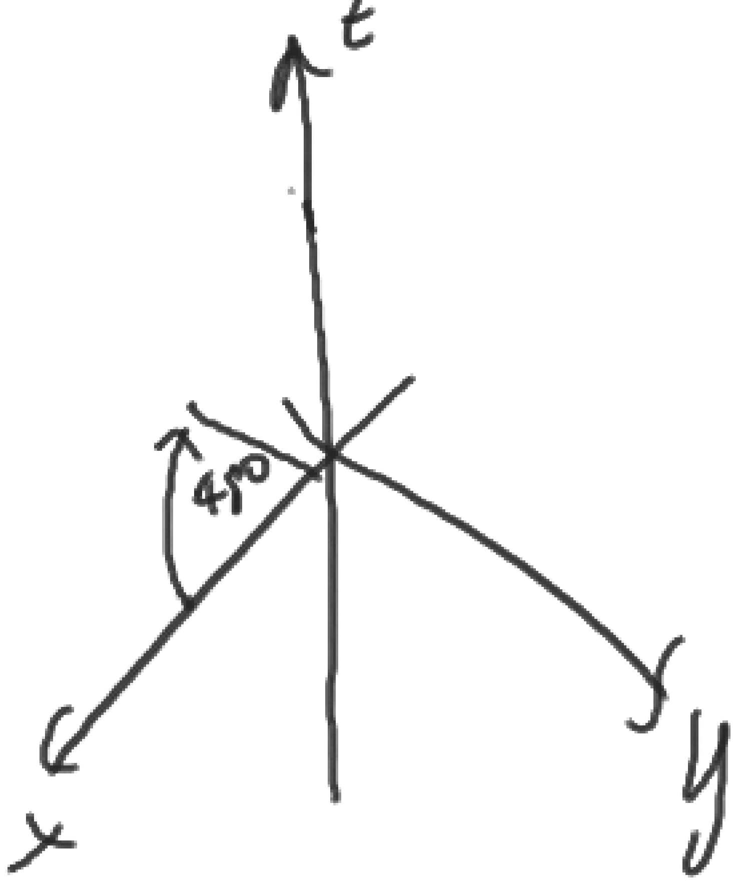
$$\sin 30^\circ = \frac{1}{2}$$

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

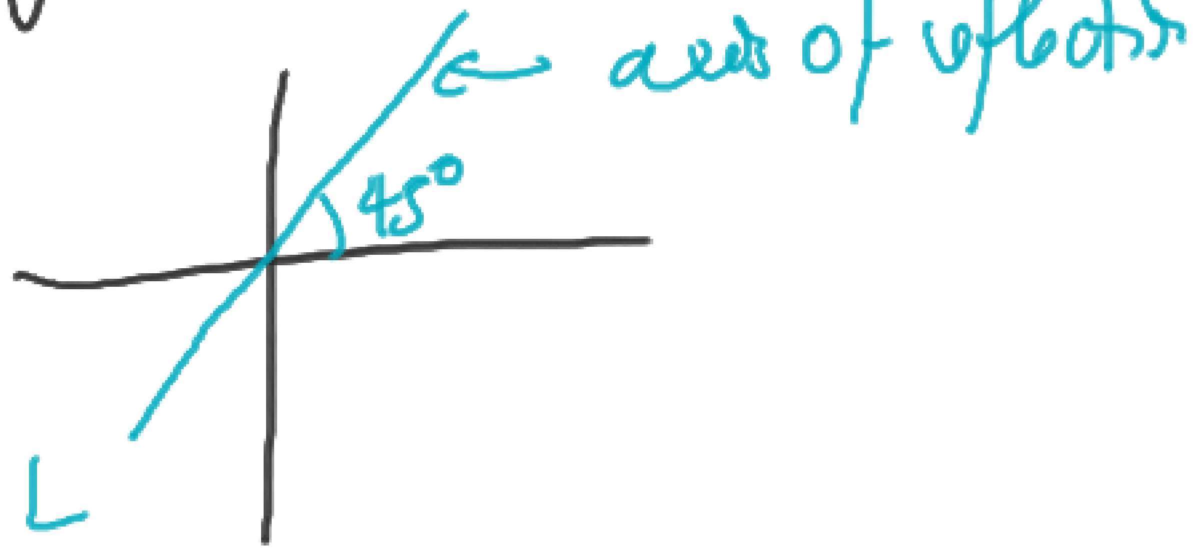
$$\begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{6}/4 & -\sqrt{2}/4 & -\sqrt{2}/2 \\ 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{6}/4 & -\sqrt{2}/4 & \sqrt{2}/2 \end{pmatrix}$$



HW (3): Find the matrix of the reflection in $\mathbb{R}^2$ by the line L obtained by rotating the x axis 45° counterclockwise about the origin.



(4) Find the matrix of the following transformation in $\mathbb{R}^3$:

First rotate in the yz plane 45° from the positive part of y-axis to the positive part of the z-axis.

Then rotate 60° in the xy plane from the positive part of the x-axis to the positive part of the y-axis.