MATH 417 2/3/2023

Exam: Next Wednesday in class

Review Monday

<u>Format</u>: 5 problems (some could have parts a, b)

no calculators, 50 min

<u>Topics</u>:
- Solving systems of linear equations (all scenarios)
- RREF – uniquely determined, columns as linear combinations of pivot columns, matrices of row operations, row equivalence

(continued on next page)

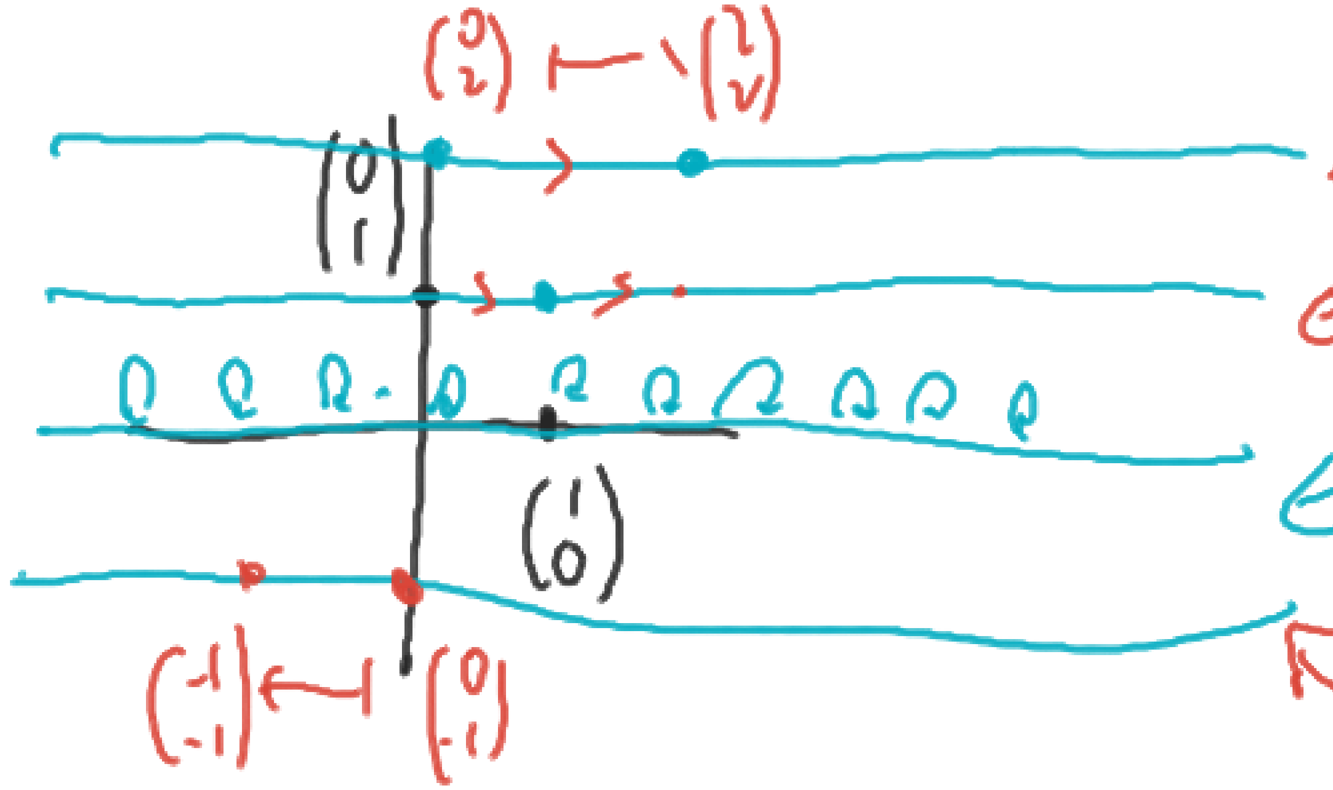
- Inverse matrix, right inverse, left inverse
- rank of a matrix
- linear transformations $\mathbb{R}^n \to \mathbb{R}^m$ matrix $\to$ formula in coordinates
  - finding the matrix of a linear transformation
  - examples covered in class up to Wednesday 2/1

(General philosophy: everything up to and including Wednesday 2/1).

More on linear transformations:

Example: Understand geometrically the linear transformation $f:\mathbb{R}^2\to\mathbb{R}^2$ corresponding to the matrix $\begin{pmatrix}1&1\\0&1\end{pmatrix}$.

Solution: $f\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ $\qquad f\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$



on the line $y=2$, every point gets shifted by 2 to the right

on the line $y=1$, every point gets shifted by 1 to the right

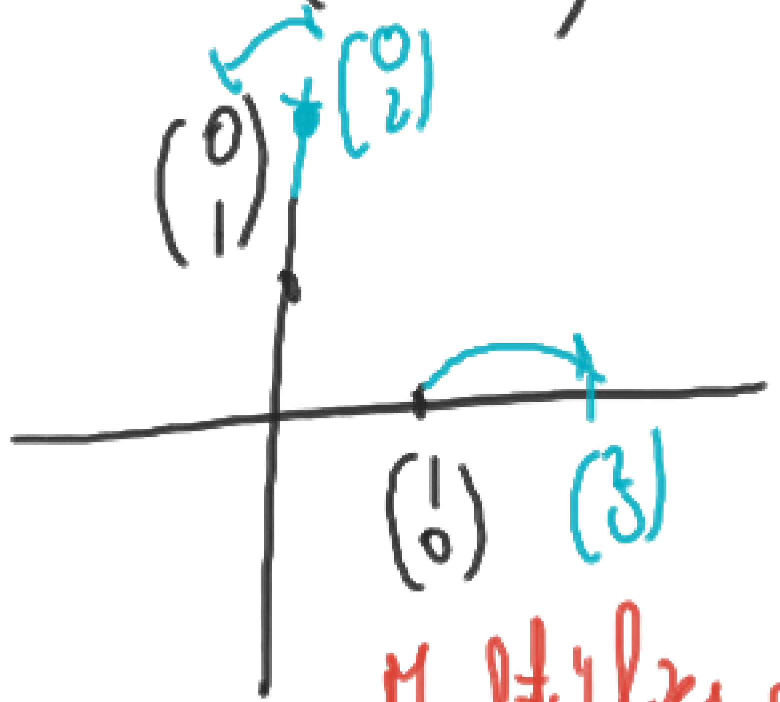
the x axis is fixed

on the line $y=-1$ every point gets shifted by 1 to the left ("-1 to the right")

On the line $y=k$, every point gets shifted by k to the right.

This mapping is called a __shearing__ (distorts distances and angles, preserves __area__)

__Easy example__: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

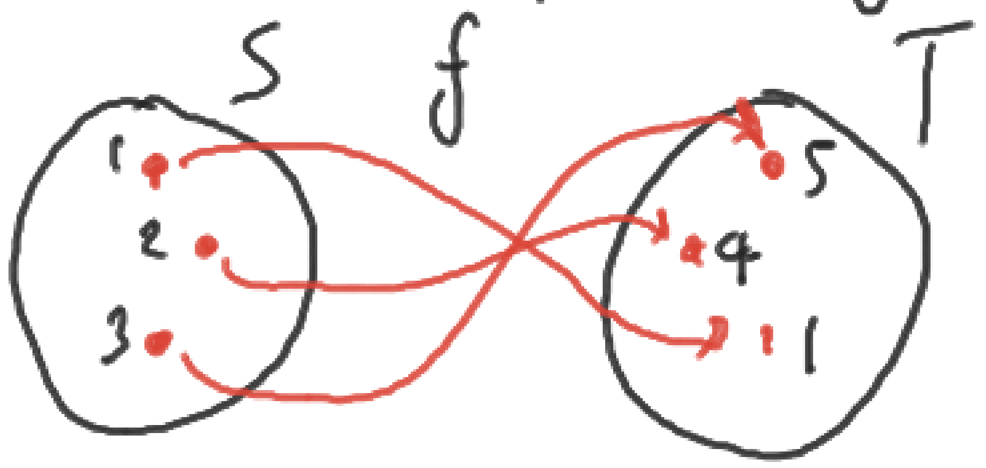


__homothety__ "magnification 2 times" while keeping the origin fixed.

Multiplies distances by 2, but preserves angles

Linear transformations $f:\mathbb{R}^n \to \mathbb{R}^m$ where $m \neq n$.

A mapping between sets $f: S \to T$ is called **bijective** if every $y \in T$ is of the form $y = f(x)$ for precisely one $x \in S$.



$f(1) = 1 \quad f(3) = 5$

$f(2) = 4$

$$\{1,2,3\} \xrightarrow[f]{\cong} \{1,4,5\}$$

Then f has an **inverse** $f^{-1}: T \to S$

$f^{-1}(y) = x$ if and only if $f(x) = y$.

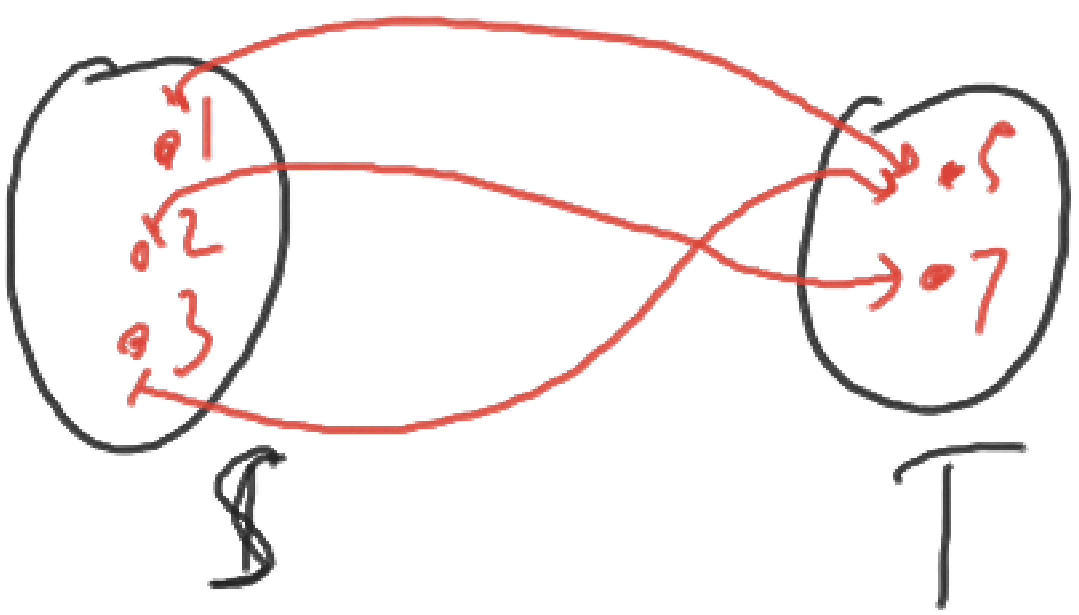
A linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ can be bijective only when $m = n$ (but we do not know without checking further)

It is bijective if and only if its matrix is invertible and then if the matrix of f is A, the matrix of f^{-1} is A^{-1}.

If $m \neq n$, this won't happen. But a mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ could be useful as "imaging".

A mapping $f : S \to T$ is called onto (surjective) when for every $y \in T$ there exist at least one $x \in S$ such that $f(x) = y$

not necessarily uniquely determined



$f(1) = 5$
$f(2) = 7$
$f(3) = 5$

Not bijective, but onto

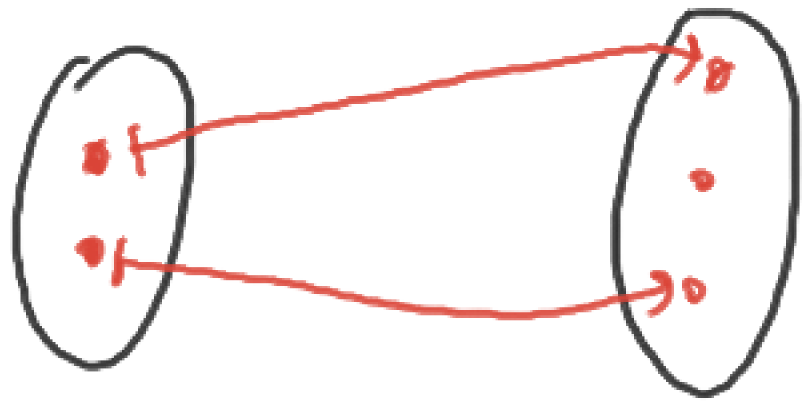
$$\{1,2,3\} \xrightarrow{f} \{5,7\}$$

If a mapping is bijective, it has a right inverse g: For $y \in T$ choose $x \in S$ with $f(x) = y$: I could choose $g(5) = 1$ $g(7) = 2$
It is true that $fg(x) = x$.

The mapping which sends every element of T to itself is called the identity Id_T $Id_T(x) = x$. $f \circ g = Id_T$.

A linear transformation f is onto if and only if it has a right inverse that is also a linear transformation.
(Its matrix will be given by the <u>right inverse</u> of the matrix of f.)

A mapping $f: S \to T$ is called <u>1-1</u> (or <u>injective</u>) if two <u>different</u> elements $x_1, x_2 \in S$ $x_1 \neq x_2$ always satisfy $f(x_1) \neq f(x_2)$



A mapping is injective if and only if it has a <u>left inverse</u> (i.e. a mapping $h: T \to S$ such that $h \circ f = Id_T$) $(f: S \to T)$

If f was a linear transformation it happens if and only if its matrix has a right inverse (which is then the matrix of the right inverse of the mapping).

(HW) (5) Decide whether the mapping below is injective, surjective, or bijective or none and find at least one right inverse, left inverse or inverse when possible:

(a) $f : \{1,2,4,5\} \to \{2,6,7,8\}$

$f(1) = 2,\ f(2) = 2,\ f(4) = 7,\ f(5) = 2$

(b) $f : \{1,2,3\} \to \{1,3,7,8\}$

$f(1) = 8,\ f(2) = 7,\ f(3) = 3$

(c) $f : \{1,2,4\} \to \{1,3\}$

$f(1) = 3 = f(2),\ f(4) = 1$

(6) Determine whether the linear transformation given by the following matrix is injective, surjective or bijective. (NO NEED to find an inverse or left, right inverse when it exists)

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 1 & 5 \end{pmatrix}$.