MATH 417 2/6/2023

Review

Systems of linear equations

Example: Find all solutions of

$$\begin{aligned} p + r + 2s + t &= 1 \\ 2p + 3r + 3s + t &= 2 \\ 3p + 4r + 5s + 2t &= 5 \end{aligned}$$

Solution:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 1 & 2 \\ 3 & 4 & 5 & 2 & 5 \end{array}\right) \begin{array}{l} \\ -2 \\ -3 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 2 \end{array}\right) \begin{array}{l} \\ \\ -1 \end{array} \qquad \left(\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right)$$

Answer: NO SOLUTION

Example: Find all solutions:

$$4p + r + 2s + 3t + 4u = 1$$
$$3p + r + s + u = 2$$

$$\left(\begin{array}{ccccc|c} 4 & 1 & 2 & 3 & 4 & 1 \\ 3 & 1 & 1 & 0 & 1 & 2 \end{array}\right) -1$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 3 & -1 \\ 3 & 1 & 1 & 0 & 1 & 2 \end{array}\right) -3$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 3 & -1 \\ 0 & 1 & -2 & -9 & -8 & 5 \end{array}\right) \nwarrow \text{RREF}$$

$$\begin{array}{ccccc} & & \uparrow & \uparrow & \uparrow \\ & & a & b & c \end{array}$$

Answer:

$$\boxed{\begin{pmatrix} p \\ r \\ s \\ t \\ u \end{pmatrix} = \begin{pmatrix} -1 - a - 3b - 3c \\ 5 + 2a + 9b + 8c \\ a \\ b \\ c \end{pmatrix}}$$

Are the following matrices row-equivalent?

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 & 7 \\ 2 & 6 & 1 & 7 & 13 \end{pmatrix} -2 \qquad B = \begin{pmatrix} 0 & 3 & 1 & 4 & 7 \\ 1 & 3 & 0 & 3 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & 4 & 7 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix} -1 \qquad \begin{pmatrix} 1 & 3 & 0 & 3 & 6 \\ 0 & 3 & 1 & 4 & 7 \end{pmatrix} 1/3$$

$$\begin{pmatrix} 1 & 3 & 1 & 4 & 7 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} -1 \qquad \begin{pmatrix} 1 & 3 & 0 & 3 & 6 \\ 0 & 1 & \frac{1}{3} & \frac{4}{3} & \frac{7}{3} \end{pmatrix} -3$$

$$\begin{pmatrix} 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{4}{3} & \frac{7}{3} \end{pmatrix}$$

RREF ← DIFFERENT → RREF

Answer: NO

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 & 7 \\ 2 & 6 & 1 & 7 & 13 \end{pmatrix}$$

Find the columns which are not linear combinations of columns to their left and express the other columns as linear combinations of these columns.

RREF: (from previous example)

$$\begin{pmatrix} 1 & 3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Not linear combinations of columns to their left: 1st and 3rd $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 7 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 7 \\ 13 \end{pmatrix} = 6 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Consider the following row operation on 4×7 matrices A:

First add 5 times the second row to the fourth row.

Then switch the first and fourth row. Find the matrix M of this operation.

$\boxed{\text{The matrix after the operation is } MA}$

It only depends on the number of rows (7 plays no role in the solution),

Do these operations in the same order on the unit matrix with the given number of rows:

4 rows →

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \qquad \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&5&0&1\end{pmatrix} \qquad \boxed{\begin{pmatrix}0&5&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix}}$$

← Answer

Find the inverse matrix (if any) of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 5 & 3 & 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 5 & 3 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \downarrow -2 \\ \downarrow -5 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -2 & -10 & -5 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \downarrow -2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right)$$

RGF

pivot on the right side of the line

NO SOLUTION

Find the inverse (if any) of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 4 & 5 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ 4 & 5 & 3 & 0 & 0 & 1 \end{array}\right) \quad -3 \quad -4$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 1 & -1 & -4 & 0 & 1 \end{array}\right) \quad -1 \quad -1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -1 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \quad 2 \quad -3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & 2 & -3 \\ 0 & 1 & 0 & -5 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} 7 & 2 & -3 \\ -5 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}}$$

Find all left respectively right inverses of the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 1 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \quad \text{(if any).}$$

No right inverses

$$\begin{pmatrix} 3 & 1 \\ 4 & 1 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \times$$

bigger

Finding right inverse to A^T

$$\left(\begin{array}{cccc|cc} 3 & 4 & 1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 2 & 0 & 1 \end{array}\right) \updownarrow$$

$$\left(\begin{array}{cccc|cc} 1 & 1 & 3 & 2 & 0 & 1 \\ 3 & 4 & 1 & 1 & 1 & 0 \end{array}\right) -3$$

$$\left(\begin{array}{cccc|cc} 1 & 1 & 3 & 2 & 0 & 1 \\ 0 & 1 & -8 & -5 & 1 & -3 \end{array}\right) -1$$

$$\left(\begin{array}{cccc|cc} 1 & 0 & 11 & 7 & -1 & 4 \\ 0 & 1 & -8 & -5 & 1 & -3 \end{array}\right)$$

$$\begin{pmatrix} -1-11a-7b & 4-11c-7d \\ 1+8a+5b & -3+8c+5d \\ a & c \\ b & d \end{pmatrix}$$

$$\boxed{\begin{pmatrix} -1-11a-7b & 1+8a+5b & a & b \\ 4-11c-7d & -3+8c+5d & c & d \end{pmatrix}}$$

Answer

Find the formula for the linear transformation

$$f : \mathbb{R}^2 \to \mathbb{R}^3$$

given by the matrix $\begin{pmatrix} 5 & 4 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$ (without using matrices)

# of columns

# of independent variables

$$\begin{pmatrix} 5 & 4 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+4y \\ 2x \\ x-y \end{pmatrix}.$$

Answer:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+4y \\ 2x \\ x-y \end{pmatrix}$$

Is $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+4y \\ y-x \end{pmatrix}$ a linear transformation? If so, find its matrix.

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

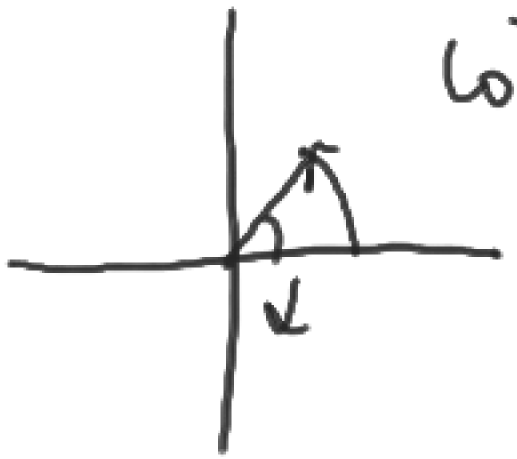
coordinates are linear combinations $\therefore$ <u>YES</u>

$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$$\boxed{A = \begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix}}$$

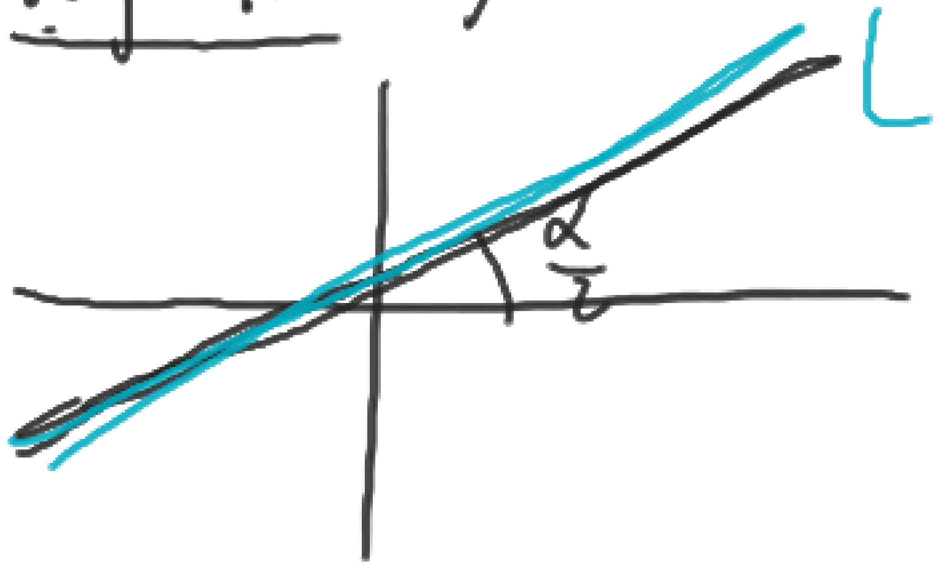
Geometric examples



rotation by α counterclockwise (around the origin in $\mathbb{R}^2$)

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

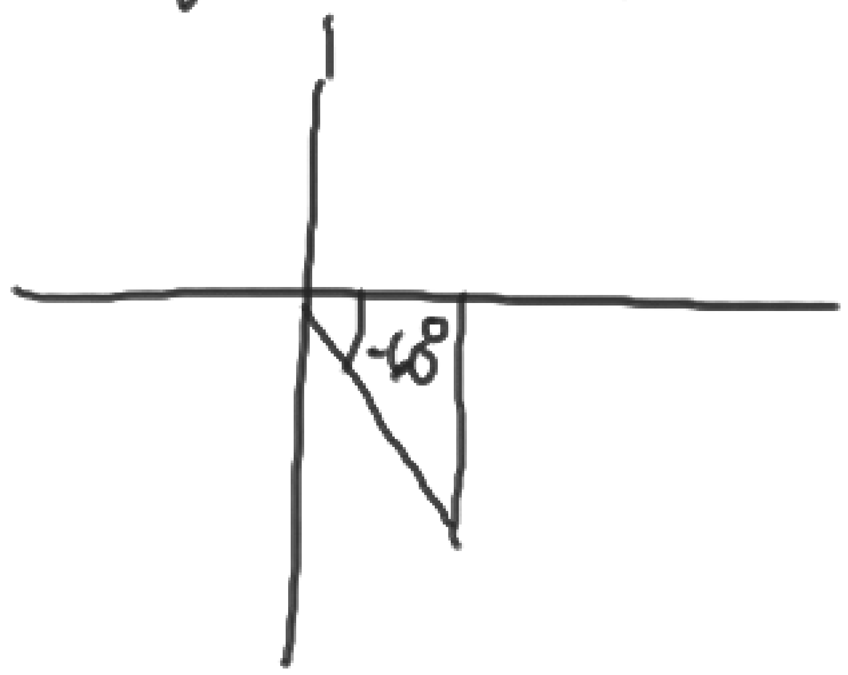
Reflection by L (x axis rotated by $\frac{\alpha}{2}$ counterclockwise



$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

? Matrix of the reflection by the line L obtained by rotating the x axis 30^0 clockwise by the origin?

$$\frac{\alpha}{2} = -30^0 \qquad \alpha = -60^0$$



$$\cos(-60^0) = \frac{1}{2}$$

$$\sin(-60^0) = -\frac{\sqrt{3}}{2}$$

$$\boxed{\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}$$

$\cos 45^\circ = \frac{1}{\sqrt{2}} = \sin 45^\circ$

$\cos 30^\circ = \frac{\sqrt{3}}{2}$ $\sin 30^\circ = \frac{1}{2}$

period 360°

$\cos(90^\circ - \alpha) = \sin\alpha$

$\cos(-\alpha) = \cos(\alpha)$

$\sin(-\alpha) = -\sin(\alpha)$

$\cos(180^\circ + \alpha) = -\cos\alpha$

$\sin(180^\circ + \alpha) = -\sin\alpha$