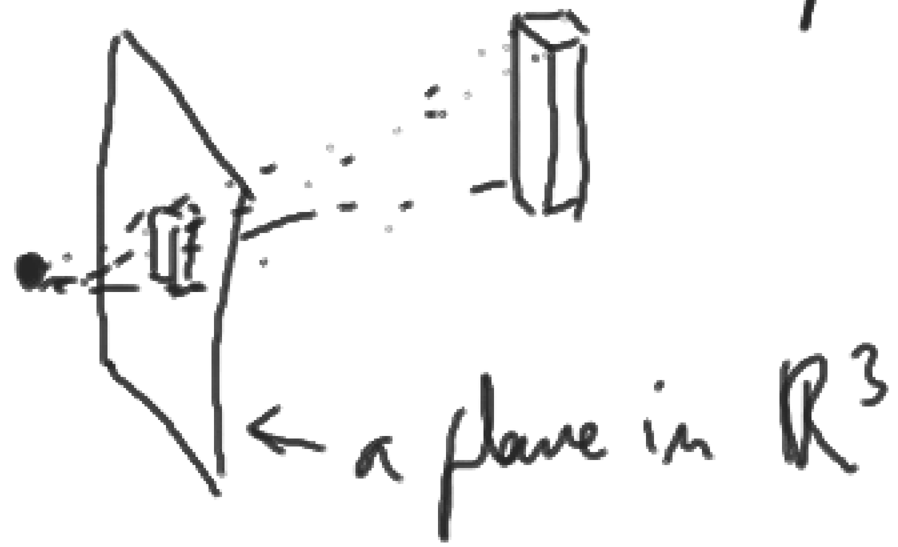
MATH 417 2/10/2023

We talked about linear transformations $f: \mathbb{R}^n \to \mathbb{R}^m$ where $m \neq n$. They cannot be bijective. But they still have geometric meaning.

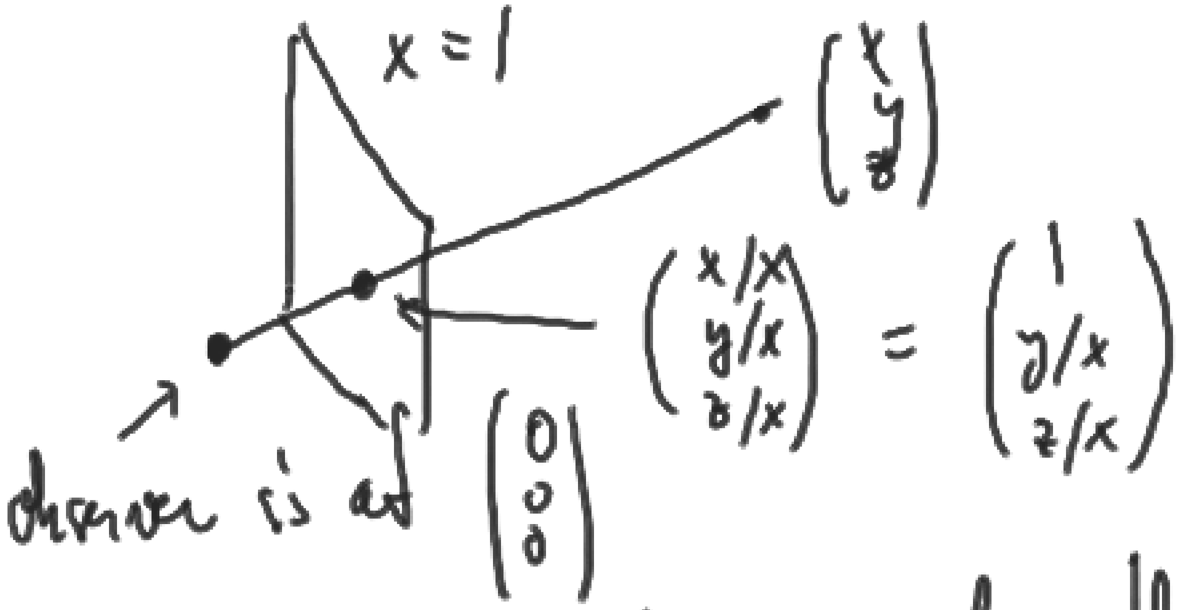
$f: \mathbb{R}^3 \to \mathbb{R}^2$ could mean a "projection", in the sense of representing a point in 3-space in a plane.

Beware: Example. The "natural perspective"




lines in space appear as lines in the plane

But the natural perspective is <u>not</u> a linear mapping




xyz-space, projecting onto the plane $x=1$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y/x \\ z/x \end{pmatrix}$$

. This is <u>not</u> linear

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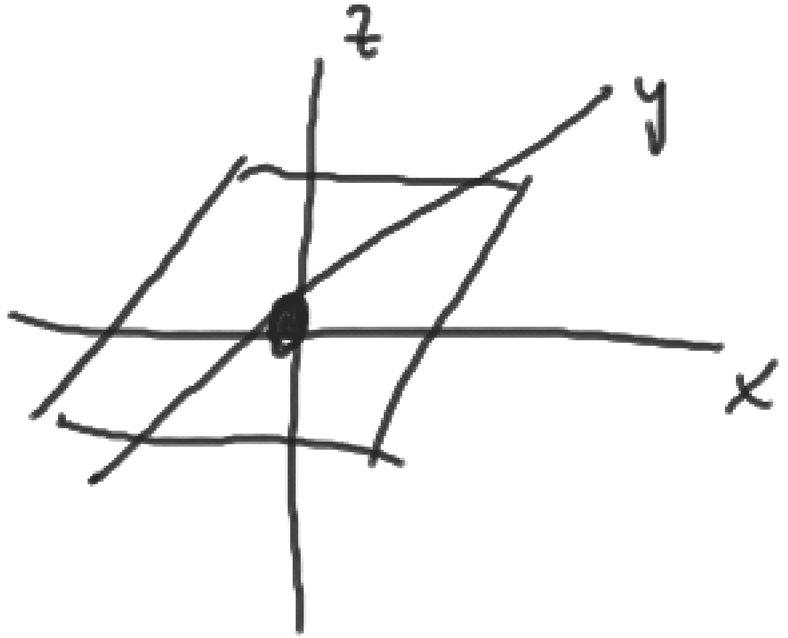
Projection which is linear would - preserve the origin
- preserve parallel lines

## Back to linear mappings:

$f : \mathbb{R}^2 \to \mathbb{R}^3$ ⟵ cannot be onto, but it maps the xy-plane onto some plane in $\mathbb{R}^3$ (possibly "slanted" in some way)




The concept we are looking for (which describes the images of linear maps) is called a vector space.

The key property of a vector space is that it has linear combinations.

A vector space is a set $V$, for any elements $v_1, \dots v_k \in V$ and real numbers $a_1, \dots a_k \in \mathbb{R}$ we have defined an element $a_1 v_1 + \dots + a_k v_k$.

## Example: Solving Differential equation

$$y'' = -ky \qquad k > 0$$

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$(\sin x)' = \cos x \qquad \sin x'' = -\sin x$

$(\cos x)' = -\sin x \qquad \cos x'' = -\cos x$

$$\left.\begin{aligned} \sin(\sqrt{k}x)'' &= -k\sin(\sqrt{k}x) \\ \cos(\sqrt{k}x)'' &= -k\cos(\sqrt{k}x) \end{aligned}\right\}$$ these are solutions.

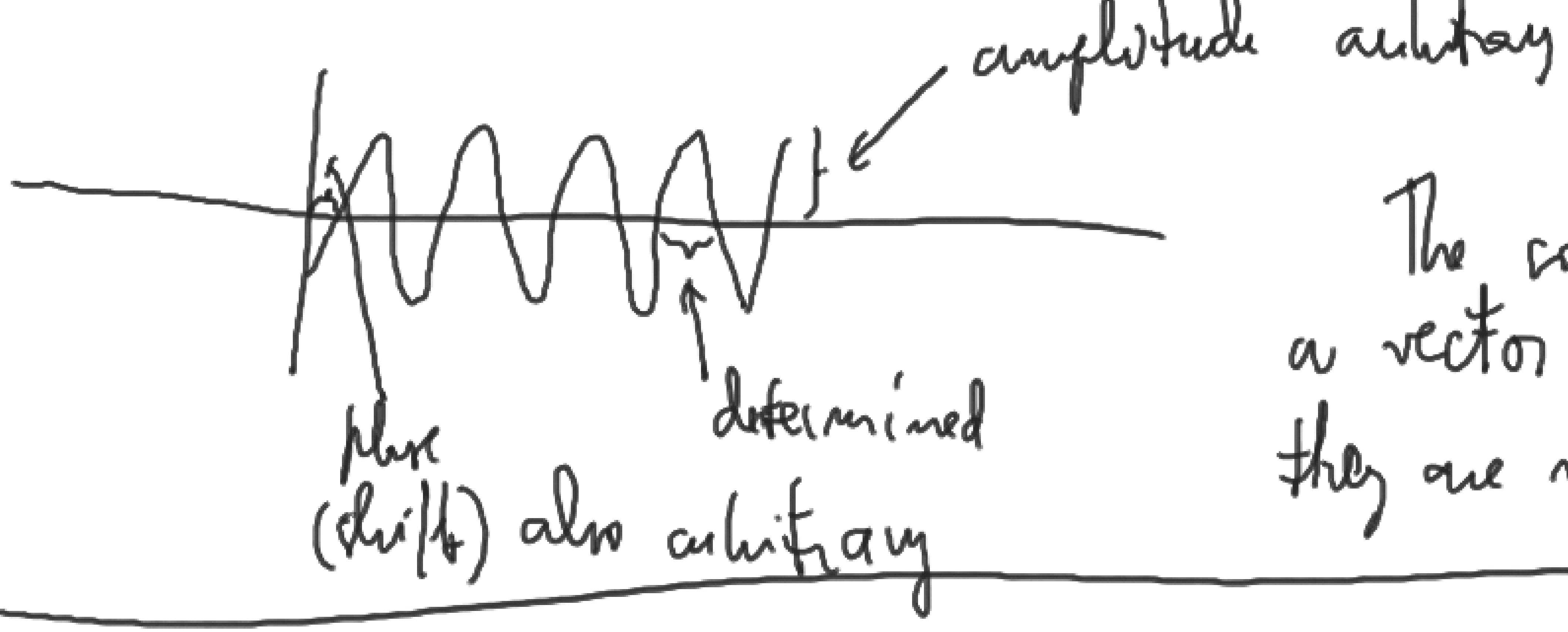
$y = \sin(\sqrt{k}x)$

$y = \cos(\sqrt{k}x)$.

any linear combination of these functions (with constant coefficients) is a solution (because it is a <u>linear</u> differential equation)

General solution: $\quad y = A\cos\sqrt{k}\,x + B\sin\sqrt{k}\,x$

where $A, B$ are constants.




The solutions form a vector space even though they are not "vectors" in $\mathbb{R}^n$.

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We specify some properties that linear combinations must satisfy. "A linear combination of linear combinations is a linear combination."

<u>Example</u>: Write

a linear combination of linear combinations

$$4(2u+3v-w) + 5(r-2u+w) + 4(s-v)$$

as a linear combination of $r, s, u, v, w$.

<u>Solution</u>: $\underline{8u} + \underline{12v} - \underline{4w} + \underline{5r} - \underline{10u} + \underline{5w} + \underline{4s} - \underline{4v}$

$$= 5r + 4s - 2u + 8v + 1w$$

$w$

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We can make linear combinations out of the following basic operations: $u+v$ (addition), $au$ $(a \in \mathbb{R})$: (multiplication of a vector by a real number)

$0 \in V$ (origin)

Basic properties (axioms):

$$0+u = u+0 = u$$

$$(u+v)+w = u+(v+w)$$

$$u+v = v+u$$

real number

0 vector

$$1\cdot u = u \qquad 0\cdot u = 0$$

$$a(b\cdot u) = (ab)\cdot u$$

$$(a+b)u = (au)+(bu)$$

$$a(u+v) = (au)+(av).$$

This is (unless I forgot something) the complete definition of a vector space.

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We are just interested in taking linear combinations and using their "usual properties", and concepts arising.

linear independence: Elements $u_1, \cdots, u_k$ of a vector space $V$ are called linearly independent if no linear combination

coefficients

$$a_1 u_1 + \cdots + a_k u_k$$

is $0$ (except when $a_1 = a_2 = \cdots a_k = 0$)

$$a_1 u_1 + \cdots + a_k u_k = 0$$

has no solution except $a_1 = a_2 = \cdots = a_k = 0$.

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Example: Are the vectors

$$\begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 5 \\ 1 \end{pmatrix} \in \mathbb{R}^4$$

linearly independent?

The test is: Make a matrix of these vectors and put it to RREF. Is every column a pivot column? (Then yes, linearly Independent). Otherwise NO.

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 4 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 2 & 5 \\ 1 & 3 & 1 \end{pmatrix} \quad -2 \quad -4 \quad -1$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \\ 0 & 2 & -1 \end{pmatrix} \quad -2 \quad 2$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ \cancel{0} & \cancel{0} & \cancel{-3} \end{pmatrix} \quad -3$$

REF

Yes, linearly independent

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<u>Note</u>: There is <u>no</u> set of more than $n$ linearly independent vectors in $\mathbb{R}^n$.

(HW) ① Are the following vectors in $\mathbb{R}^5$ linearly independent?

$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \\ 5 \\ 5 \end{pmatrix}$$

② Are the following vectors in $\mathbb{R}^4$ linearly independent?

$$\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 4 \\ 1 \end{pmatrix}.$$