MATH 417 2/13/2023

If V is a vector space (= a set where linear combinations are defined and have reasonable properties) $v_1, \dots v_n \in V$, we say that $v_1, \dots v_n$ are <u>linearly independent</u> if $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$ only occurs when $a_1 = a_2 = \dots = a_n = 0$.

We say that $v_1, \dots v_n$ <u>span</u> V (or <u>generate</u>) if every element $w \in V$ is a linear combination of $v_1, \dots, v_n$:

$w = a_1 v_1 + \dots + a_n v_n$ for some choice of $a_1, \dots a_n \in \mathbb{R}$.

Example: Do the vectors

$$\begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}2\\4\\2\\2\end{pmatrix}, \begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\2\\1\end{pmatrix}, \begin{pmatrix}4\\7\\4\\4\end{pmatrix}$$

span $\mathbb{R}^4$?

Solution:

$$A = \begin{pmatrix}1&2&1&2&4\\2&4&1&3&7\\1&2&2&2&4\\1&2&1&1&4\end{pmatrix} \begin{matrix}\\ )\cdot(-2)\\ -1\\ -1\end{matrix}$$

We could do column operations to get into RCEF and see if we have 4 pivots. rank $A = 4$

But we can also test rank with row operations (we proved that the number of pivots is the same)

$$\begin{pmatrix}1&2&1&2&4\\0&0&-1&-1&-1\\0&0&1&0&0\\0&0&0&-1&0\end{pmatrix} \begin{matrix}\\ \\ )\,1\\ \end{matrix}$$

$$\begin{pmatrix}1&2&1&2&4\\0&0&-1&-1&-1\\0&0&0&-1&-1\\0&0&0&-1&0\end{pmatrix} \begin{matrix}\\ \\ \\ )-1\end{matrix}$$

$$\begin{pmatrix}\boxed{1}&2&1&2&4\\0&0&\boxed{-1}&-1&-1\\0&0&0&\boxed{-1}&-1\\0&0&0&0&\boxed{1}\end{pmatrix}$$

REF 4 pivots.

Answer YES it spans.

n vectors $v_1, \dots, v_n \in V$ are called a *basis* if they are linearly independent and also span V.

Question: Does every vector space V have a basis?

Answer: Yes, but not necessarily finite.

Example: All real functions with domain $\mathbb{R}$:

$$f : \mathbb{R} \to \mathbb{R}$$

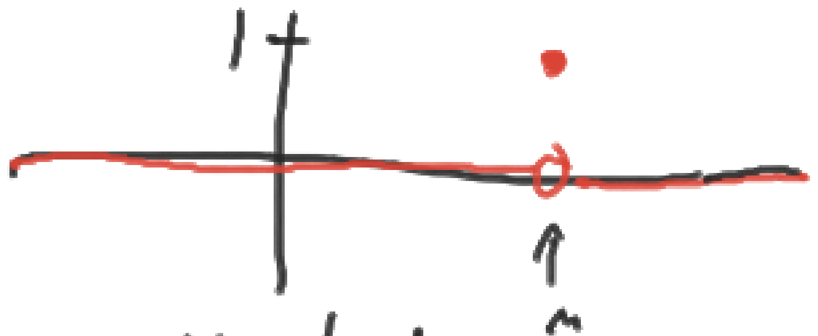
$$(f+g)(x) := f(x) + g(x)$$

$$(af)(x) := af(x).$$

Define functions $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = 0 \qquad x \neq n$$

$$f_n(n) = 1.$$



These are linearly independent. So V cannot have a finite basis.

We say that a vector space V is <u>finite-dimensional</u> if it has a finite basis.

If V has a finite basis $v_1, \dots, v_n$ then every element $w \in V$ can be expressed as a linear combination of $v_1, \dots v_n$ in a uniquely determined way:

$$w = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

uniquely determined real numbers

coordinates of w with respect to the basis $v_1, \dots v_n$

If you also had

$$w = b_1 v_1 + b_2 v_2 + \dots + b_n v_n$$

($a_1 \neq b_1$ or $a_2 \neq b_2$ or ... or $a_n \neq b_n$)

subtracting,

$$0 = (a_1 - b_1) v_1 + (a_2 - b_2) v_2 + \dots + (a_n - b_n) v_n$$

This would contradict $v_1, \dots, v_n$ being linearly independent.

Example: (a) Do the vectors $\begin{pmatrix}1\\2\\-1\end{pmatrix}\begin{pmatrix}2\\5\\1\end{pmatrix}\begin{pmatrix}3\\7\\1\end{pmatrix}$ form a basis of $\mathbb{R}^3$?

(b) If so, find the coordinates of $\begin{pmatrix}2\\2\\5\end{pmatrix}$ with respect to this basis.

Solution: For (a), the RREF has to be the identity matrix. For (b), need to solve $\begin{pmatrix}2\\2\\5\end{pmatrix} = x\begin{pmatrix}1\\2\\-1\end{pmatrix} + y\begin{pmatrix}2\\5\\1\end{pmatrix} + z\begin{pmatrix}3\\7\\1\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}1&2&3&2\\2&5&7&2\\-1&1&1&5\end{array}\right) \quad \left(\begin{array}{ccc|c}1&0&0&-7\\0&1&0&-15\\0&0&1&13\end{array}\right)$$

$$\left(\begin{array}{ccc|c}1&2&3&2\\0&1&1&-2\\0&3&4&7\end{array}\right)$$

$$\left(\begin{array}{ccc|c}1&0&1&6\\0&1&1&-2\\0&0&1&13\end{array}\right)$$

(a) Yes (b) Coordinates: $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-7\\-15\\13\end{pmatrix}$ ✓

If we have the coordinates of a vector w with respect to some basis $B: u_1, \dots, u_n$, how can we find the coordinates of w with respect to another basis $C: v_1, \dots, v_m$

it will turn out this has to be the same number!

This can be accomplished by a matrix.

If we make a column vector out of the B-coordinates of w, we call this the coordinate vector w_B.

There is a matrix M called the base change matrix such that

$$v_C = M v_B$$

This matrix is invertible and hence square. This is why any two bases of a vector space V have the same number of elements, called the dimension $\dim(V)$

HW 1 Do the vectors

$$\begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\7\\1\\4\end{pmatrix}, \begin{pmatrix}4\\11\\0\\7\end{pmatrix}$$

span $\mathbb{R}^4$?

2 a Do the vectors B: $\begin{pmatrix}1\\1\\-1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\\3\end{pmatrix}, \begin{pmatrix}3\\2\\0\\1\end{pmatrix}, \begin{pmatrix}4\\3\\-1\\4\end{pmatrix}$ form a basis of $\mathbb{R}^4$?

b If so, find the coordinate vector w_B where

$$w = \begin{pmatrix}7\\1\\2\\5\end{pmatrix}$$