MATH 417 2/15/2023

$B: u_1, \ldots, u_n$ basis of a vector space V

$w \in V$: $w = a_1 u_1 + \cdots + a_n u_n$, $a_1, \ldots, a_n \in \mathbb{R}$ uniquely determined

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = w_B$$

coordinate vector of the vector w with respect to the basis B

Suppose $C: v_1, \ldots, v_n$ is another basis. How do we find w_C from w_B? There is a matrix ${}_C M_B$ (called a base change matrix) such that $w_C = {}_C M_B \, w_B$.

Same principle: The columns of ${}_C M_B$ are the C-coordinate vectors of the elements of the basis B.

$B: u_1, \dots, u_n \qquad C: v_1, \dots v_n$

$$\boxed{{}_C M_B = \begin{pmatrix} (u_1)_C & (u_2)_C & \cdots & (u_n)_C \end{pmatrix}}$$

$${}_C M_B \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{=(u_1)_B} = (u_1)_C$$

$${}_C M_B \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{(u_k)_B} = (u_k)_C$$

k row

In the general setting: ${}_B M_B = I$

${}_C M_C = I$

$$\left.\begin{aligned} {}_C M_B \; {}_B M_C &= {}_C M_C = I \\ {}_B M_C \; {}_C M_B &= {}_B M_B = I \end{aligned}\right\}$$

${}_B M_C$, ${}_C M_B$ are inverse to each other, so they are square matrices.

So this is why any two bases of the same vector space have the same number of elements.

How do we find ${}_C M_B$ for two bases of $\mathbb{R}^n$?

$$\left(C \mid B \right) \sim \left(I \mid {}_C M_B \right)$$

(If we get a RREF not in this form, there is a mistake. Either numerical error, or B, or C is not a basis.)

__Example__: Find the base change matrix from the basis

$$B = \begin{pmatrix}2\\3\\4\end{pmatrix}, \begin{pmatrix}4\\1\\1\end{pmatrix}, \begin{pmatrix}6\\4\\4\end{pmatrix}$$

to the basis

$$C = \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}.$$

__Solution__:

$$\left(\begin{array}{ccc|ccc}1&1&2&2&4&6\\1&2&3&3&1&4\\2&-1&0&4&1&4\end{array}\right) \begin{matrix}-1\\-2\end{matrix}$$

Check: $C \cdot {}_CM_B = B$ ✓

$$\left(\begin{array}{ccc|ccc}1&1&2&2&4&6\\0&1&1&1&-3&-2\\0&-3&-4&0&-7&-8\end{array}\right) \begin{matrix}-1\\3\end{matrix} \qquad \left(\begin{array}{ccc|ccc}1&0&1&1&7&8\\0&1&1&1&-3&-2\\0&0&1&-3&16&14\end{array}\right) \begin{matrix}-1\\-1\end{matrix}$$

$$\boxed{{}_CM_B = \begin{pmatrix}4&-9&-6\\4&-19&-16\\-3&16&14\end{pmatrix}}$$

$$\left(\begin{array}{ccc|ccc}1&0&1&1&7&8\\0&1&1&1&-3&-2\\0&0&-1&3&-16&-14\end{array}\right) \cdot(-1) \qquad \left(\begin{array}{ccc|ccc}1&0&0&4&-9&-6\\0&1&0&4&-19&-16\\0&0&1&-3&16&14\end{array}\right)$$

In $\mathbb{R}^n$, we have one basis E: $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$.

This is the standard basis. If we have another basis C of $\mathbb{R}^n$, then

$${}_E M_C = C$$

$${}_C M_E = C^{-1}$$

There are other examples of vector spaces in $\mathbb{R}^n$.

Suppose I have a vector space V, and some elements $v_1, \ldots, v_k \in V$.

Suppose $v_1, \ldots v_k$ do not necessarily span V. Then I can still take the set of all linear combinations of $v_1, \ldots, v_k$ in V. This is denoted by $\langle v_1, \ldots, v_k \rangle = \{a_1 v_1 + \cdots + a_k v_k \mid a_1, \ldots, a_k \in \mathbb{R}\}$

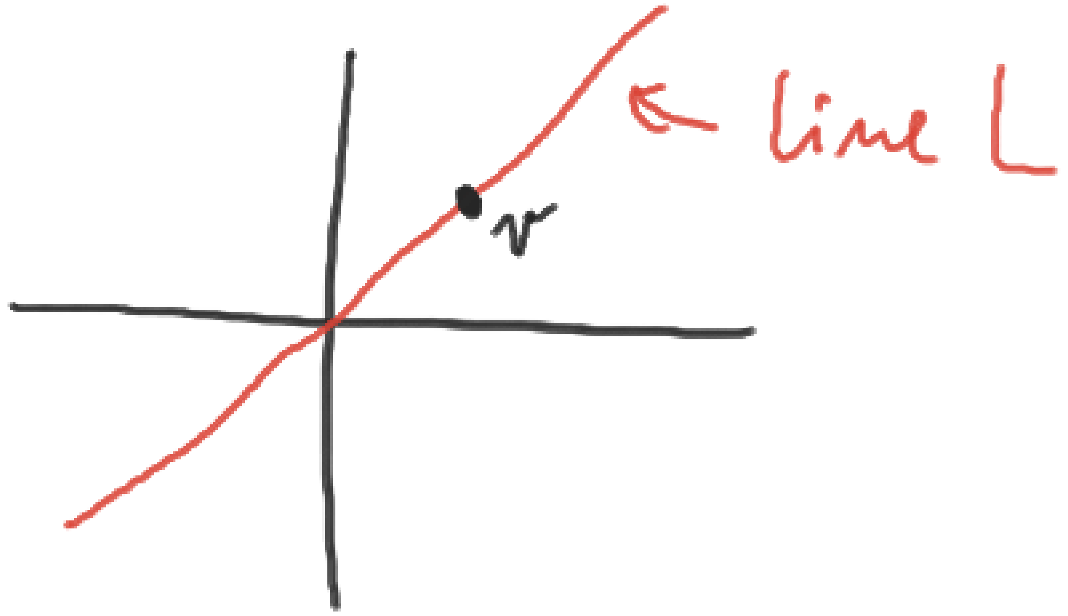
The span of $v_1, \ldots, v_k$ or the vector subspace generated by $v_1, \ldots v_k$.

Examples from geometry: The subspace in $\mathbb{R}^n$ generated by 0 vectors is just the origin 0.

The subspace in $\mathbb{R}^n$ generated by one non-zero vector is a line through the origin.

The vector subspace in $\mathbb{R}^n$ generated by two linearly independent vectors is a plane.

A subspace of $\mathbb{R}^n$ has no "standard basis" in general.



$v \in L \qquad v \neq 0$

$L = \langle v \rangle$

a basis of L

How do we find a basis of a subspace of $\mathbb{R}^n$?

there will be many solutions!

Example: Find a basis of the span

$$\left\langle \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^4$$

Answer: $\begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 2 \end{pmatrix}$

Solution: We just need to find the pivot columns

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ -1 & 3 & -4 & -1 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 \end{pmatrix} \begin{matrix} +1 \\ -2 \\ -1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 5 & -5 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 5 & -5 & 0 \\ 0 & -3 & 3 & 0 \end{pmatrix} \begin{matrix} +5 \\ -3 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -3 \end{pmatrix} \; 3/5$$

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot

HW (3) Find the base change matrix ${}_C M_B$ from the basis

$$B: \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \end{pmatrix} \quad \text{of } \mathbb{R}^4$$

to the basis

$$C: \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$$

4 Find a basis of

$$\left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ 3 \\ 2 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^5.$$