MATH 417 2/17/2023

"Express a line of the form $3x + 4y = 0$ as the set of all tv where v is some fixed vector in $\mathbb{R}^2$ and $t \in \mathbb{R}$."

($v = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ e.g.)

Slightly more complicated for a plane in $\mathbb{R}^3$. We can do this in any dimension, for a system of linear equations with right-hand side 0. (homogeneous linear equations).

General fact: The set of all solutions to $Av = 0$ where A is an $(m \times n)$-matrix and $v \in \mathbb{R}^n$ is a $\underbrace{\text{vector subspace of } \mathbb{R}^n}_{\text{The solution space}}$.

Example: Find the basis of the solution space in $\mathbb{R}^5$ of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 3 & 1 & 1 \end{pmatrix}.$$

Solution: We are asking for all $v \in \mathbb{R}^5$ such that $Av = 0$. Put A into RREF:

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 3 & 1 & 1 \end{pmatrix} \; -2$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 & -1 \end{pmatrix} \; -1$$

$$\begin{array}{c} \begin{pmatrix} 1 & 2 & 0 & 5 & 2 \\ 0 & 0 & 1 & -3 & -1 \end{pmatrix} \\ \begin{array}{ccccc} & \uparrow & & \uparrow & \uparrow \\ & A & & B & C \end{array} \end{array}$$

General solution:

$$\begin{pmatrix} -2A-5B-2C \\ A \\ 3B+C \\ B \\ C \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\text{Basis: } \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}} \; \checkmark$$

Example: Find a basis of the $\overbrace{\text{vector subspace}}^{\text{plane through the origin}}$ of $\mathbb{R}^3$ given by the equation

$$3x + 2y + z = 0$$

Solution: Same as in the last problem.

$$\begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \cdot \frac{1}{3}$$

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{array}{ccc} & \uparrow & \uparrow \\ & A & B \end{array}$$

$$\begin{pmatrix} -\frac{2}{3}A - \frac{1}{3}B \\ A \\ B \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{\text{Answer: } \begin{pmatrix} -2/3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1/3 \\ 0 \\ 1 \end{pmatrix}}$$

The general method for finding the solution space of an $m \times n$ matrix A:

1. Put A into RREF
2. Find the general solution of $Av = 0$ in parametric form and then plug in the unit matrix (or any matrix with maximal rank) into the parameters.

The dimension of the solution space of A is:

$$\boxed{n - \operatorname{rank}(A)}$$

We studied a linear transformation $f: V \to W$ where V is a finite-dimensional vector space. With f, two additional vector spaces arise:

the set of all elements $f(v)$ where $v \in V$

vector subspace

$$\mathrm{Im}(f) = \{ f(v) \mid v \in V \} \subseteq W$$

Image

the set of all element v of V such that $f(v) = 0$.

$$\mathrm{Ker}(f) = \{ v \in V \mid f(v) = 0 \} \subseteq V$$

Kernel (= solution space)

$$\boxed{\dim \mathrm{Ker}(f) + \dim \mathrm{Im}(f) = \dim V}$$

Back to base change. Bases are <u>really needed</u> for vector spaces which aren't just $\mathbb{R}^n$. How do we find the base change matrix from a basis $B: u_1, \dots, u_k$ to another basis $C: v_1, \dots, v_k$ of the same vector space V? The columns of the base change matrix ${}_C M_B$ are the C-coordinate vectors of the element of B:

$$ {}_C M_B = \begin{pmatrix} (u_1)_C & (u_2)_C & \cdots & (u_k)_C \end{pmatrix} $$

Example: A vector subspace $V \subseteq \mathbb{R}^4$ has two bases:

vector subspace (also used for "subset")

$$B: \begin{pmatrix} 4 \\ 10 \\ 7 \\ 4 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 11 \\ 8 \\ 4 \end{pmatrix}$$

$$C: \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 4 \\ 2 \end{pmatrix}$$

Solution:

$$(C \mid B) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 3 & 4 \\ 2 & 3 & 5 & 10 & 7 & 11 \\ 1 & 2 & 4 & 7 & 4 & 8 \\ 1 & 1 & 2 & 4 & 3 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{c|c} I & {}_C M_B \\ 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 3 & 4 \\ 2 & 3 & 5 & 10 & 7 & 11 \\ 1 & 2 & 4 & 7 & 4 & 8 \\ 1 & 1 & 2 & 4 & 3 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 4 & 3 & 4 \\ 0 & 1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 3 & 1 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\boxed{C \, {}_C M_B = B}$$

$$\boxed{{}_C M_B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}}$$

Answer

Homework: ① Find the basis of the solution space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 & 6 \\ 1 & 3 & 4 & 6 & 7 \end{pmatrix}$$

② A vector subspace $V \subseteq \mathbb{R}^4$ has bases

$$B: \begin{pmatrix} 3 \\ 2 \\ 4 \\ 3 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \\ 7 \\ 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 8 \\ 2 \end{pmatrix} \qquad C: \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

Find the base change matrix ${}_C M_B$ from B to C.