MATH 417 2/20/2023

<u>Example</u>: Let V be the vector space $\underbrace{\text{generated}}_{\text{spanned}}$ by the functions e^x, e^{-x}. Consider the basis $A: e^x, e^{-x}$. Is $B: \cosh x, \sinh x$ a basis of the same vector space V? If so, find the base change matrices ${}_A M_B$ (B to A) and ${}_B M_A$ (from A to B).

Solution: $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$A: e^x, e^{-x}$

$B: \cosh x, \sinh x$

$$(\cosh x)_A = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$(\sinh x)_A = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$$

$${}_A M_B = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

from B to A

$${}_B M_A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}^{-1}$$

from A to B

$$\det({}_A M_B) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$${}_B M_A = -2 \cdot \begin{pmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}}$$

Slight digression: A general formula for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$

"generic" method – assume the numbers are such that no div. by 0 occurs

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \cdot 1/a$$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right) -c$$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -c/a & 1 \end{array}\right) \cdot \frac{a}{ad-bc}$$

$$\left(\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) -\frac{b}{a}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\underbrace{\frac{1}{a} + \frac{bc}{a(ad-bc)}}_{\frac{\cancel{a}d}{\cancel{a}(ad-bc)}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The determinant:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

inverse A^{-1} exist. if and only if $\det A \neq 0$

linear transformations between arbitrary vector spaces

$$f : V \to W$$

Defining properties of a linear transformation ("preserves linear combinations")

$$f(av) = af(v) \qquad v \in V, \; a \in \mathbb{R}$$

$$f(v_1 + v_2) = f(v_1) + f(v_2).$$

Suppose we have a basis $B : v_1, \dots v_n$ of V and a basis $C : w_1, \dots w_m$ of W. The matrix ${}_C f_B$ of the linear transformation f with respect to the basis B of the domain and the basis C of the codomain is defined by the formula

$$\underbrace{{}_C f_B}\, v_B = f(v)_C$$

$\begin{pmatrix} m \times n \\ \text{matrix} \end{pmatrix} \cdot \begin{pmatrix} n \\ \text{vector} \end{pmatrix} = \begin{pmatrix} m \\ \text{vector} \end{pmatrix}$

the matrix of the linear transformation of f with respect to B and C

coordinate vector of v with respect to B

$v_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} : v = a_1 v_1 + \dots + a_n v_n$

$f(v) = b_1 w_1 + \dots + b_m w_m$

$f(v)_C = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

To get the columns of ${}_C f_B$, take the C-coordinate vectors of f applied to the elements of B.

"like the base change matrix, except you apply f".

$${}_C f_B = \begin{pmatrix} f(v_1)_C & \cdots & f(v_n)_C \end{pmatrix} \quad (m \times n \text{ matrix})$$

Example: In $\mathbb{R}^3$, consider a plane V through the origin with basis

$$B: \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$$

and the plane W through the origin with basis $C: \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}$.

Now let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be given by the matrix $A = \begin{pmatrix}3&4&2\\2&3&2\\1&2&2\end{pmatrix}$. $(f(v) = Av)$.

Does f map the plane V to the plane W? If so, find ${}_C f_B$.

Solution: $(C \mid f(B)) \sim \left(\begin{array}{c|c} I & {}_C f_B \\ 0 & 0 \end{array}\right)$ ← RREF. If RREF looks different, then f does not map V to W.

← maybe, maybe not.

$$f(B) = \begin{pmatrix}3&4&2\\2&3&2\\1&2&2\end{pmatrix}\begin{pmatrix}1&1\\1&0\\0&1\end{pmatrix} = \begin{pmatrix}7&5\\5&4\\3&3\end{pmatrix}$$

$$\left(\begin{array}{cc|cc}2&1&7&5\\1&1&5&4\\0&1&3&3\end{array}\right) \sim \left(\begin{array}{cc|cc}1&1&5&4\\2&1&7&5\\0&1&3&3\end{array}\right) \sim \left(\begin{array}{cc|cc}1&1&5&4\\0&-1&-3&-3\\0&1&3&3\end{array}\right) \sim \left(\begin{array}{cc|cc}1&1&5&4\\0&1&3&3\\0&1&3&3\end{array}\right) \sim \left(\begin{array}{cc|cc}1&0&\boxed{2}&\boxed{1}\\0&1&\boxed{3}&\boxed{3}\\ \cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}\end{array}\right)$$

${}_C f_B$

Answer:

$${}_C f_B = \begin{pmatrix}2&1\\3&3\end{pmatrix}$$

(HW) In today's problems, we are given a linear transformation $f: \mathbb{R}^n \to \mathbb{R}^m$ in terms of a matrix A $(f(v) = Av)$.
We are also given a basis B of a vector subspace $V \subseteq \mathbb{R}^n$ and a basis C of a vector subspace $W \subseteq \mathbb{R}^m$. <u>Question</u>: Does f map V to W and if so, find its matrix ${}_C f_B$.

(1) $f: \mathbb{R}^4 \to \mathbb{R}^3$

$$A = \begin{pmatrix} 3 & 2 & 5 & 4 \\ 2 & 2 & 3 & 2 \\ 4 & 6 & 5 & 2 \end{pmatrix}$$

$B: \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}$

$V = \langle B \rangle \quad W = \langle C \rangle$

$C: \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

(2) $f : \mathbb{R}^3 \to \mathbb{R}^4$ $V = \langle B \rangle$ $W = \langle C \rangle$

B : $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

C : $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \\ 5 \end{pmatrix}$

Will be due Monday March 6
10 AM