MATH 417 02/22/2023

Example: Consider the linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(v) = Av$ where $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix}$. Does f map the vector subspace (plane) V with basis $B: \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ to the vector subspace (plane) with basis $C: \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$? If so, find ${}_C f_B$.

Solution: $f(B) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 6 & 9 \\ 6 & 10 \end{pmatrix}$

$\boxed{\text{Answer: } f \text{ does not map } V \text{ into } W.}$

$$(C|f(B)) = \left(\begin{array}{cc|cc} 1 & 1 & 5 & 8 \\ 1 & 2 & 6 & 9 \\ 2 & 3 & 6 & 10 \end{array}\right) \begin{matrix} -1 \\ -2 \end{matrix} \quad \left(\begin{array}{cc|cc} 1 & 1 & 5 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & -7 \end{array}\right) \leftarrow \text{REF}$$

not equivalent to $\left(\begin{array}{c|c} I & {}_C f_B \\ 0 & 0 \end{array}\right)$

$$\left(\begin{array}{cc|cc} 1 & 1 & 5 & 8 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -4 & -6 \end{array}\right) -1$$

If $f: U \to V$ and $g: V \to W$ are linear transformations and if B is a basis of U, C is a basis of V, and D is a basis of W then:

$$_D(g \circ f)_B = {}_Dg_C \circ {}_Cf_B .$$

In particular, if f is bijective (invertible), then

$$_B(f^{-1})_C = ({}_Cf_B)^{-1}$$

Two vector spaces of the same dimension $\underbrace{\text{look exactly alike.}}_{\text{mathematical word: isomorphic}}$

A bijective linear transformation $f: V \to W$ is called an <u>isomorphism</u>.

"Any $\underbrace{\text{vector space}}_{\text{finite-dimensional}}$ V is exactly like $\mathbb{R}^n$ $(n = \dim V)$

as long as we pick a basis B of V."

When we say that a linear transformation $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is given by $f(v) = Av$ where A is an $m \times n$ matrix, this really means

$$A = {}_{E_m}f_{E_n}$$

E_n is the standard basis $\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \dots \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$.

What about the base change matrix ${}_CM_B$ (B, C are bases of the same vector space V)? The identity on a set S is the mapping $Id_S : S \to S$ given by $Id_S(x) = x$.

The base-change matrix is the matrix of the identity: $\boxed{{}_CM_B = {}_C(Id_V)_B}$.

This tells us how to transform the matrix of a linear transformation when we change the basis in the domain and/or the codomain.

<u>Example</u>: Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ which has matrix $A = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$ with respect to the standard basis in the domain and codomain: $A = {}_E f_E$. What is the matrix ${}_C f_B$ where $B: \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $C: \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$?

<u>Solution</u>:

$$ {}_C f_B = \underbrace{{}_C \mathrm{Id}_E}_{C^{-1}} \cdot \underbrace{{}_E f_E}_{\underbrace{\begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}}_{A}} \cdot \underbrace{{}_E \mathrm{Id}_B}_{\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}} $$

$$ \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 21 & 17 \\ 29 & 23 \end{pmatrix} $$

$$ \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = C^{-1} $$

$$ \det \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = 3 - 2 = 1 $$

$$ \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 21 & 17 \\ 29 & 23 \end{pmatrix} = \boxed{\begin{pmatrix} 34 & 28 \\ -13 & -31 \end{pmatrix}} $$

← <u>Answer</u>

The reason we study linear transformations is that they are the first approximation of non-linear transformations. Let V, W be finite-dimensional vector spaces, and we have some mapping $f: V \to W$.

If $\underbrace{x \in V}_{\text{"point"}}$, $\underbrace{v \in V}_{\text{"vector"}}$. The derivative of f by v at x (if one exists) is

$$(\partial_v f)_x = \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h} \qquad (h \in \mathbb{R})$$

If we fix a point D, we can define $Df_x : V \to W$

$$Df_x(v) = (\partial_v f)_x .$$

Then Df_x is a linear transformation called the $\overbrace{\text{total differential}}^{\text{linearization}}$ of f at x.

We have a linear approximation (more precisely, "affine approximation")

$$\boxed{f(y) \sim f(x) + Df_x \cdot (y - x).}$$

__Example__: Find the linear approximation of $f:\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s^2 - 2st \\ t^2 + s \end{pmatrix}$$

at $\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

__Solution__: The matrix of the total differential with respect to the standard bases is the matrix of partial derivatives (Jacobi matrix):

$$A = \begin{pmatrix} \partial f_1/\partial s & \partial f_1/\partial t \\ \partial f_2/\partial s & \partial f_2/\partial t \end{pmatrix} = \begin{pmatrix} 2s - 2t & -2s \\ 1 & 2t \end{pmatrix}$$

$@ \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$$

$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

$f_1 = s^2 - 2st$

$f_2 = t^2 + s$

$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\boxed{f\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}}$$ __Answer__

HW 3 Find the linear approximation of $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s^3 - 2st^2 \\ s^2 + t^2 + 2st \end{pmatrix}$

at $\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

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4 Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation where ${}_E f_E = \begin{pmatrix} -1 & 4 \\ 2 & 3 \end{pmatrix}$. Find ${}_C f_B$ where $B: \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $C: \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.