MATH 417 3/6/2023

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad \det(A) = \sum_{\substack{\text{permutation} \\ \sigma: \{1,\dots,n\} \to \{1,\dots,n\}}} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

A permutation $\sigma: \{1, \dots, n\} \to \{1, \dots, n\}$ is a bijective mapping.
(there are $n!$ of them)

A permutation can be encoded by the sequence of numbers

$$(\sigma(1), \sigma(2), \dots, \sigma(n))$$

($\underbrace{\text{Each number}}_{\text{in } \{1,\dots,n\}}$ occurs exactly once) The sign of the "identity permutation" $(1, 2, 3, \dots n)$ is $+$.

Switching exactly two numbers in the sequence switches the sign.

Permutation with the sign + are called <u>even</u>
permutations with the sign − are called <u>odd</u>

How do we know this is consistent?

<u>Example</u>:

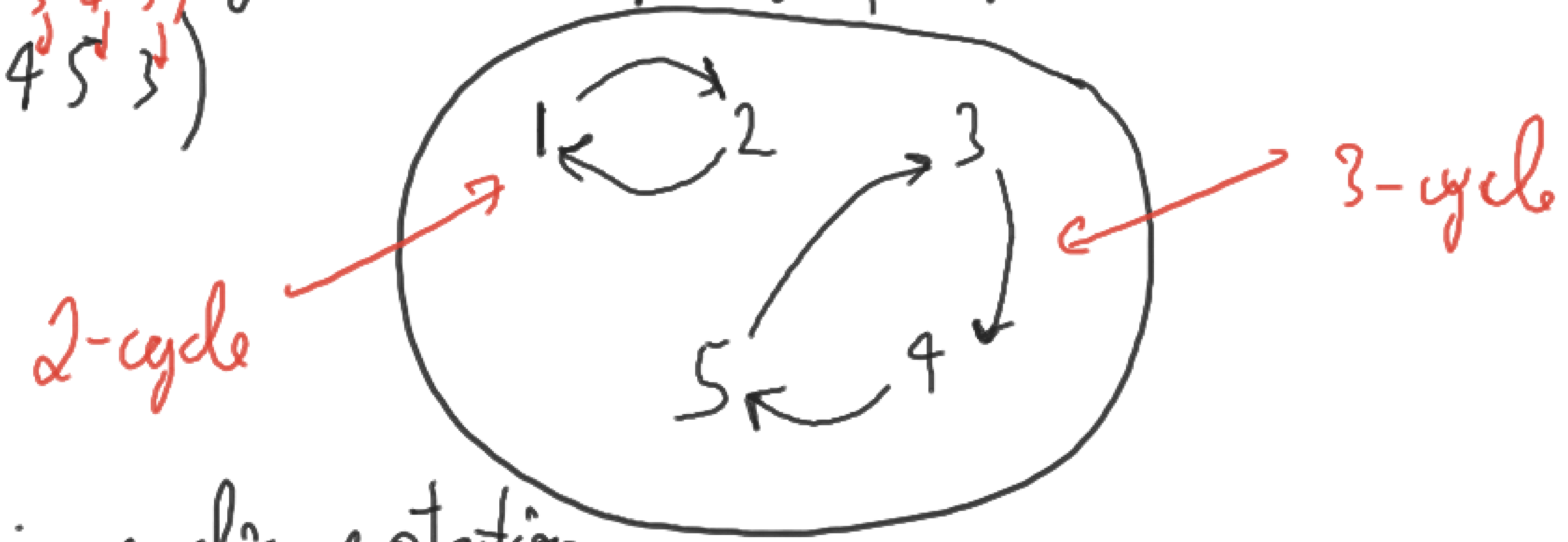
$$\left.\begin{array}{l}(2\,1\,4\,5\,3)\\(1\,2\,4\,5\,3)\\(1\,2\,3\,5\,4)\\(1\,2\,3\,4\,5)\end{array}\right\} \quad 3 \text{ switches} \Rightarrow \text{odd}$$

$$\left.\begin{array}{l}(2\,1\,4\,5\,3)\\(2\,1\,4\,3\,5)\\(2\,1\,3\,4\,5)\\(3\,1\,2\,4\,5)\\(1\,3\,2\,4\,5)\\(1\,2\,3\,4\,5)\end{array}\right\} \quad 5 \text{ switches} \Rightarrow \text{odd}$$

How do we know the even or odd permutation definition is consistent?

We need to find a way to see the parity of a permutation right away.

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 1 & 4 & 5 & 3\end{pmatrix}$$



The permutation in <u>cyclic notation</u>

$$(12)(345)$$

= Variability in cyclic notation:
same as

$$(21)(453)$$

A 2-cycle is an odd permutation ("RED")

An even-length cycle is an odd permutation

An odd-length cycle is an even permutation

To tell whether a permutation is even or odd, count its <u>even-length cycles</u> only. Even number of even-length cycles ⇒ even

Odd number of even-length cycles ⇒ odd

one even-length cycle

odd permutation

Example: Let σ be the permutation given by the sequence

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \end{array}$$
$$(9\,3\,2\,8\,5\,4\,7\,6\,1)$$

(a) Write the permutation σ in cyclic notation

(b) Use the cyclic notation to tell if σ is even or odd

fixed points = 1-cycles

Solution:

$$\boxed{(1\,9)(2\,3)(4\,8\,6)(5)(7)}$$

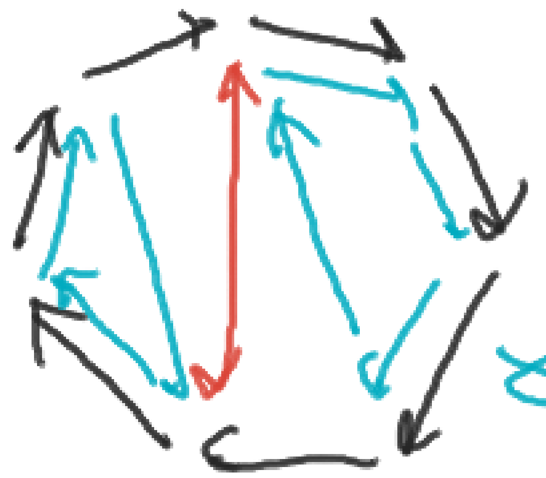
Answer for (a)

2 even-length cycles

$\boxed{\text{Even permutation}}$ Answer (b)

$(4\,8\,6) = (8\,6\,4) \neq (8\,4\,6)$

We need to show that if we perform a single switch the the sign (or parity) of a permutation changes.



If the switch happens within one cycle, the cycle splits into two cycles

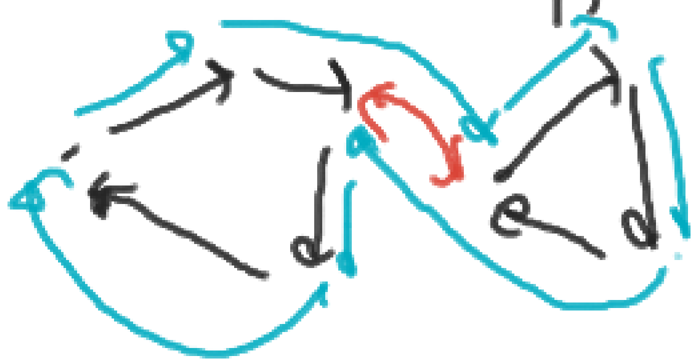
after composing with the switch (first do black move then red move)

odd cycle $\to$ even cycle, odd cycle

even cycle $\to$ even cycle, even cycle

even cycle $\to$ odd cycle, odd cycle

If your switch goes between two different cycles, it will join them (reverse move)



Example: A permutation σ in cyclic notation is

$$(12)(3568)(47)$$ ← (3 even-length cycles, odd permutation)

Express this permutation in sequence notation.

Solution.

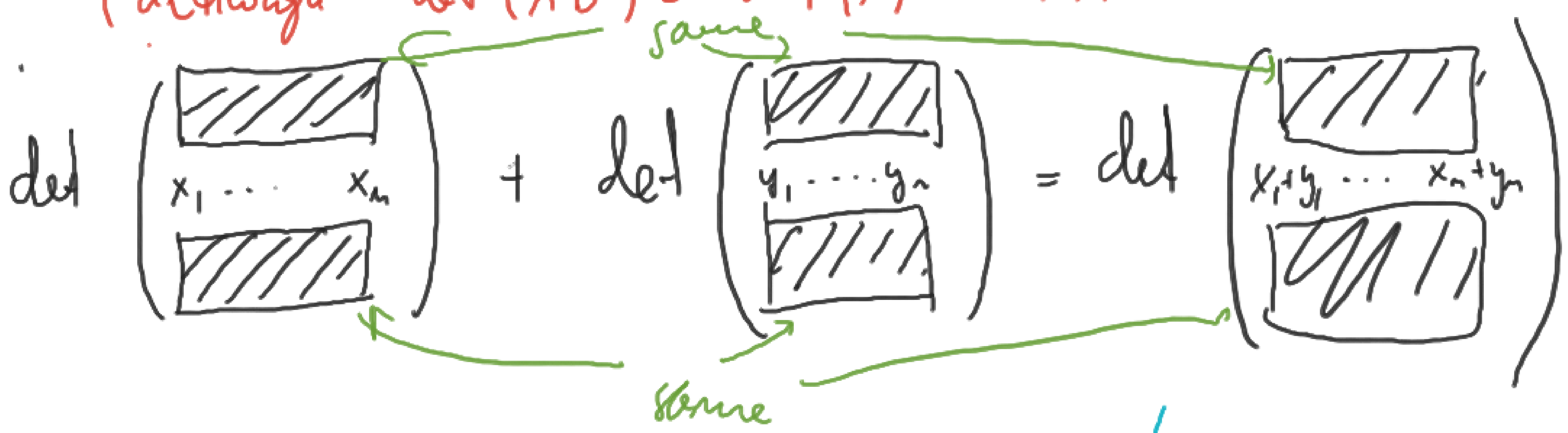
$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \end{array}$$ ← Where do these numbers go?

$$\boxed{(2\ 1\ 5\ 7\ 6\ 8\ 4\ 3)}$$ ← **Answer**

Back to determinants. They are "multilinear"

$\det(A+B)$ is usually <u>not</u> the same as $\det(A)+\det(B)$

(although $\det(AB) = \det(A)\det(B)$).



$$\det\begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 2 & 2 & 1 \\ 3 & 4 & 1 & 5 \\ 2 & 1 & 1 & 4 \end{pmatrix} + \det\begin{pmatrix} 2 & 1 & 4 & 3 \\ 5 & 6 & 1 & 7 \\ 3 & 4 & 1 & 5 \\ 2 & 1 & 1 & 4 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 4 & 3 \\ 6 & 8 & 3 & 8 \\ 3 & 4 & 1 & 5 \\ 2 & 1 & 1 & 4 \end{pmatrix}$$

Same thing for columns.

Also, if you multiply one row or column by a number, the determinant is also multiplied by that number:

$$4 \det \begin{pmatrix} \boxed{\begin{matrix}1\\2\\3\end{matrix}} & \begin{matrix}2\\1\\1\end{matrix} & \begin{matrix}3\\1\\1\end{matrix} \end{pmatrix} = \det \begin{pmatrix} \boxed{\begin{matrix}4\\8\\12\end{matrix}} & \begin{matrix}2\\1\\1\end{matrix} & \begin{matrix}3\\1\\1\end{matrix} \end{pmatrix}$$

HW ① A permutation σ is given in sequence notation as

$(2\,4\,9\,6\,7\,8\,1\,5\,3)$.

ⓐ Write the permutation σ in cycle notation

ⓑ Use the cycle notation to determine if σ is even or odd.

② A permutation σ is given in cyclic notation as

$$(1\,4\,7\,3)(2\,6\,5).$$

Express σ in sequence notation.