MATH 417 3/8/2023

Determinants are multilinear, which means linear in one row or column, keeping everything else the same.

$$\det \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{A} = \underbrace{\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & 0 \cdots & 0 \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}}_{a_{i1} \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 1 & 0 \cdots & 0 \\ & & \end{pmatrix}} + \cdots + \underbrace{\det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \cdots 0 & a_{in} \\ & & \end{pmatrix}}_{a_{in} \det \begin{pmatrix} & & \\ 0 & \cdots 0 & 1 \\ & & \end{pmatrix}}$$

$A_{ij}$ is obtained from $A$ by deleting the $i$'th row and $j$'th column

$$= (-1)^{i+1} \det A_{i1} + \cdots + (-1)^{i+j} \det A_{ij} + \cdots$$

$A_{ij}$ – sometimes called a minor (deleting $i$th row and $j$th column from $A$)

Row expansion formula: <u>choose</u> $i$th row:

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \dots + (-1)^{i+n} a_{in} \det A_{in}$$

Column expansion formula: <u>choose</u> $j$th column:

$$\det A = (-1)^{1+j} a_{1j} \cdot \det A_{1j} + (-1)^{2+j} a_{2j} \cdot \det A_{2j} + \dots (-1)^{n+j} a_{nj} \cdot \det A_{nj}$$

Useful for sparse matrices (have many 0s).

<u>Example</u>:

$$\det \underbrace{\begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0 \quad 3} \\ 0 & 1 & 2 & 4 \end{pmatrix}}_{A} = (-1)^{7} \cdot 3 \det A_{34} =$$

3+4

← 3rd row

4th column

step 1

$$-3 \det \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = -6 \det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} =$$

$$= -6(2-1) = \boxed{-6}.$$

step 2

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$$\det \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 1 \cdot \det \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} - 1 \det \begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix} =$$

$$= 4 - 2 = 2$$

Fundamental fact (why the signs?):

**If $A$ has two identical rows (or two identical columns) then $\det A = 0$**

$$\det \begin{pmatrix} 2 & 1 & 4 & 5 \\ 3 & 2 & 4 & 1 \\ 2 & 1 & 4 & 5 \\ 6 & 7 & 0 & -5 \end{pmatrix} = 0$$

opposite signs

$$\det \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix} = 0$$

$$\underbrace{1\cdot 1\cdot 1 - 1\cdot 1\cdot 1}_{} \underbrace{- 2\cdot 2\cdot 1 + 2\cdot 2\cdot 1}_{} \underbrace{+ 3\cdot 2\cdot 1 - 3\cdot 1\cdot 2}_{} = 0$$

$$\det A = \sum_{\sigma:\{1,\dots,n\}\to\{1,\dots,n\}} \operatorname{sign}(\sigma)\, \underbrace{a_{1\sigma(1)} \cdots a_{n\sigma(n)}}_{\text{products come in pairs with opposite signs so the sum is } 0.} = 0$$

permutations

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & & a_{nn} \end{pmatrix}$$

This means that adding a multiple of one row (or column) to another does not change the determinant

$$\det\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{pmatrix} \overset{-2}{} = \det\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{pmatrix} - 2\underbrace{\det\begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{pmatrix}}_{0}$$

two same rows

$$= \det\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 & 3 & 1 \\ -2 & -4 & -6 & -2 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & -5 & -1 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{pmatrix}$$

This elementary row operation does not change the determinant!

$$\det\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 7 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{pmatrix} \quad -2,\ -3,\ -4$$

Go to REF using Gauss elimination
(if you switch two rows, the sign changes!)

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & -5 & -1 \\ 0 & -4 & -8 & -2 \\ 0 & -7 & -11 & -3 \end{pmatrix} \quad 1$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -13 & -3 \\ 0 & -4 & -8 & -2 \\ 0 & -7 & -11 & -3 \end{pmatrix} \quad -4,\ -7$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -13 & -3 \\ 0 & 0 & 44 & 10 \\ 0 & 0 & 80 & 18 \end{pmatrix} \quad -\frac{80}{44} \quad \sim \quad \begin{pmatrix} 1 & 2 & 3 & 1 \\ & -1 & -13 & -3 \\ 0 & & 44 & 10 \\ & & 0 & 18 - \frac{800}{44} \end{pmatrix}$$

$$-\frac{8}{44}$$

$$(-1)\cdot\det\begin{vmatrix} 44 & 10 \\ 80 & 11 \end{vmatrix} =$$

$$= -1\,(792 - 800) = \underline{\underline{8}}$$

Calculate $\det \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 4 & 1 \\ 1 & 4 & 5 & 2 \\ 2 & 8 & 11 & 10 \end{pmatrix}$ by Gauss elimination

Solution:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 4 & 1 \\ 1 & 4 & 5 & 2 \\ 2 & 8 & 11 & 10 \end{pmatrix} \quad \ominus$$

$$\begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 5 & 2 \\ 2 & 8 & 11 & 10 \end{pmatrix} \quad -1 \quad -2$$

$$\begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 8 \end{pmatrix} \quad -1 \quad -2$$

$$\begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad -1$$

$$\begin{pmatrix} 1 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

In REF, product of diag. terms $-4$

→ one switch: Answer: $\boxed{4}$

HW: (3) Calculate via row or column expansion:

$$\det\begin{pmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(4) Calculate via Gauss elimination:

$$\det\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 1 \\ 3 & 5 & 4 & 5 \\ 5 & 8 & 6 & 10 \end{pmatrix}$$