MATH 417 3/10/2023

Exam Wednesday in class - same format

Topics :
- Vector spaces, linearly independent set, spanning set
- basis of a vector space : Subspaces of $\mathbb{R}^n$, column space, solution space
- base change matrix, coordinate vectors
- the matrix of a linear transformation with respect to a basis of the domain and the codomain
- determinant : definition, even, odd permutations, row/column expansion
- calculating determinants by Gauss elimination

Example:

$$\det\begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 \\ 3 & 2 & 3 & 1 \\ 6 & 4 & 7 & 1 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 \\ 3 & 2 & 3 & 1 \\ 6 & 4 & 7 & 1 \end{pmatrix} \ominus$$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 3 & 1 \\ 6 & 4 & 7 & 1 \end{pmatrix} \quad -2 \quad -3 \quad -6$$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -2 & -5 \\ 0 & -1 & -3 & -11 \\ 0 & -2 & -5 & -23 \end{pmatrix} \quad -1 \quad -2$$

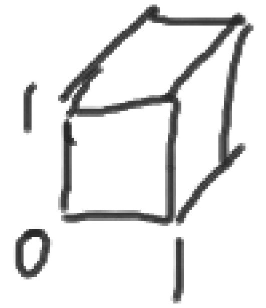
$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & -1 & -13 \end{pmatrix} \quad -1$$

$$\begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

$$-(-7) = \boxed{7}$$

Answer

The determinant has a relationship with volume.

$\underbrace{\text{volume}}_{\text{measure}}$ in $\mathbb{R}^n$:  unit cube has volume 1.

Suppose a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ has matrix A (with respect to the standard bases in the domain and codomain).

Eudoxos: shearing does not change volume.

A linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ always affects $\underbrace{\text{volume}}_{n\text{-dim.}}$ by multiplying it by a constant $|\det(A)|$.

In particular, $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ unit cube is transformed by A into an n-dimensional parallelepiped P determined by the column vectors $\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix}$. $\Big\}$ $\mathrm{vol}(P) = |\det A|$

Example: Compute the area of the parallelogram in $\mathbb{R}^2$ determined by the vectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Answer: $\left|\det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\right| = |-1| = \underline{\underline{1}}$ Answer.

Example: Find the volume of the parallelepiped in $\mathbb{R}^3$ determined by the vectors

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Solution:

row expansion in the first row

$$\det \begin{pmatrix} 2 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 3 \end{pmatrix} = 2 \cdot \underset{(3\cdot 3 - 4 \cdot 1)}{5} - 3 \cdot \underset{(1 \cdot 3 - 1 \cdot 1)}{2} + 1 \cdot \underset{(1 \cdot 4 - 1 \cdot 3)}{1} = 10 - 6 + 1 = \underline{\underline{5}}$$

The sign of the determinant means <u>orientation</u>.

In the physical world (as opposed to imaging), we cannot implement linear transformations with a negative determinant.

<u>Example</u>: Calculate

$$\det\begin{pmatrix} 2 & 1 & 3 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 4 & 2 & 6 & 7 \end{pmatrix}$$

<u>Solution</u>:

$$\begin{pmatrix} 2 & 1 & 3 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 4 & 2 & 6 & 7 \end{pmatrix} \updownarrow \ominus$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ 1 & 0 & 1 & 2 \\ 4 & 2 & 6 & 7 \end{pmatrix} \quad -2,\ -1,\ -4$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & -2 & 3 \end{pmatrix} \quad -1,\ -2$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \updownarrow \ominus$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Answer</u>:

$$\boxed{0}$$

$\boxed{\text{The determinant } \det(A) \text{ of an } n\times n \text{ matrix } A \text{ is nonzero if and only if } \operatorname{rank}(A) = n.}$

number of pivots in RREF

Recall: this is if and only if A^{-1} exists.

$$\boxed{\det(A\cdot B) = \det(A)\det(B).}$$

A, B $n\times n$ matrices

(If one of them has rank $< n$ then so does AB, so both sides are 0. Any $n\times n$ matrix of rank n is a product of elementary row operation matrices.)

HW 5 Find the volume of the parallelepiped in $\mathbb{R}^3$ spanned by the vectors

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 5 \\ 10 \end{pmatrix}.$$

6 Determine the orientation ($+$ or $-$) of the linear transformation $f(v) = Av : \mathbb{R}^4 \to \mathbb{R}^4$

$$A = \begin{pmatrix} 2 & 3 & 1 & 2 \\ 3 & 4 & 1 & 1 \\ 5 & 8 & 1 & 5 \\ 10 & 15 & 3 & 1 \end{pmatrix}.$$

$$s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix} + u\begin{pmatrix} 6 \\ 5 \\ 10 \end{pmatrix}$$

$$0 \le s, t, u \le 1$$

Comment