MATH 417 3/13/2023

<u>Review</u>

Find a basis of

$$\left\langle \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\2\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}3\\4\\5\end{pmatrix} \right\rangle$$

$$\begin{pmatrix}1&2&1&0&3\\1&2&2&-1&4\\2&4&1&1&5\end{pmatrix} \begin{matrix}-1\\-2\end{matrix} \sim \begin{pmatrix}1&2&1&0&3\\0&0&1&-1&1\\0&0&-1&1&-1\end{pmatrix} \begin{matrix}-1\\1\end{matrix} \sim \begin{pmatrix}1&2&0&1&2\\0&0&1&-1&1\\0&0&0&0&0\end{pmatrix}$$

RREF

(REF would suffice)

pivot columns 1,3

Answer: Basis: $\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$.

Find a basis of the space of solutions of the equation

$Ax = 0$ $\qquad x \in \mathbb{R}^5$

where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ 1 & 2 & 2 & -1 & 4 \\ 2 & 4 & 1 & 1 & 5 \end{pmatrix}$$

(same matrix as in last problem to save time)

$$A \sim \begin{array}{c} \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccc} & \uparrow & & \uparrow & \uparrow \\ & s & & t & u \end{array} \end{array}$$

$$\begin{pmatrix} -2s - t - 2u \\ s \\ t - u \\ t \\ u \end{pmatrix}$$

$$\begin{pmatrix} s \\ t \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

<u>Answer</u>:

$$\boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}$$

Let V be the space of $x \in \mathbb{R}^5$ where $Ax = 0$, $A = \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ 1 & 2 & 2 & -1 & 4 \\ 2 & 4 & 1 & 1 & 5 \end{pmatrix}$.

We just constructed a basis B: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

Is C: $\begin{pmatrix} -4 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$ a basis of V and if so, find the base change matrix ${}_C M_B$ from B to C.

$$\boxed{C \cdot {}_C M_B = B}$$

$$(C|B) = \left(\begin{array}{ccc|ccc} -4 & -3 & 5 & -2 & -1 & -2 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ -4 & -3 & 5 & -2 & -1 & -2 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} 4 \\ 1 \\ -1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 2 & -1 & -2 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$ rearrange rows

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -3 & 1 & 2 & -1 & -2 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{array}\right) \begin{array}{l} -1 \\ 3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 & 1 & -1 \end{array}\right) \cdot -1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) 1$$

$$\left(\begin{array}{ccc|ccc} 1 & & & 0 & -1 & 1 \\ & 1 & & -1 & 0 & 1 \\ & & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\boxed{\text{Answer } {}_C M_B = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}}$$

Two subspaces of $\mathbb{R}^4$ are $V = \langle \underbrace{\begin{pmatrix}2\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\\1\end{pmatrix}}_{B} \rangle$, $W = \langle \underbrace{\begin{pmatrix}3\\1\\2\\2\end{pmatrix}, \begin{pmatrix}2\\1\\0\\1\end{pmatrix}}_{C} \rangle$

We have a linear map $f: V \to W$ given by $f(x) = Ax$

$$A = \begin{pmatrix} -3 & 5 & 1 & 1 \\ -2 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix}.$$

Does f map $f: V \to W$? If so, find the matrix ${}_C f_B$.

$(C \mid f(B))$

$$\begin{pmatrix} -3 & 5 & 1 & 1 \\ -2 & 4 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \left(\begin{array}{c|c} 2 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{array}\right) = \begin{pmatrix} 1 & 5 \\ 0 & 2 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc} 3 & 2 & 1 & 5 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 1 & 3 \end{array}\right)$$

Answer: Yes,

$${}_C f_B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 2 \\ 3 & 2 & 1 & 5 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 1 & 3 \end{array}\right) \begin{matrix} 2-3 \\ -2 \\ -2 \end{matrix} \quad \left(\begin{array}{cc|cc} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -2 & 2 & -2 \\ 0 & -1 & 1 & -1 \end{array}\right) 2\cdot(-1) \quad \left(\begin{array}{cc|cc} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 2 & -2 \\ 0 & -1 & 1 & -1 \end{array}\right) \begin{matrix} 5-1 \\ 2 \\ 1 \end{matrix} \quad \left(\begin{array}{cc|cc} 1 & 0 & \boxed{1} & \boxed{1} \\ 0 & 1 & \boxed{-1} & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Convert the permutation in sequence notation $\begin{array}{c} \color{red}{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9} \\ (2\ 3\ 9\ 4\ 6\ 5\ 1\ 7\ 8) \end{array}$ to cycle notation. Is it even or odd?

$$\boxed{(1\,2\,3\,9\,8\,7)(4)(5\,6)}$$ ← Answer

Two even cycles ∴ The permutation is <u>even</u>.

Convert the permutation in cycle notation $(1\,4\,5)(2\,3\,6)(7\,9)(8)$ to sequence notation.

$$\begin{array}{c} \color{red}{1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9} \\ \boxed{(4\ 3\ 6\ 5\ 1\ 2\ 9\ 8\ 7)} \end{array}$$ Answer

Calculate:

$$\det \begin{array}{c c} & \begin{array}{ccccc} & 2 & & & \end{array} \\ 3 \rightarrow & \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix} \end{array} = -4 \cdot \det \begin{pmatrix} 5 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix} = -8 \det \begin{pmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$$

column expansion
$2+3=5$ odd
$-$

column expansion
$4+4$ even $+$

column expansion
$3+3$ even $+$

$$= -24 \det \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix} = \underline{\underline{-120}}$$

Answer

Calculate: $\det \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 3 & 0 \\ 3 & 4 & 6 & 1 \\ 4 & 6 & 7 & 0 \end{pmatrix}$ -1 -3 -4

the take product of diag. terms

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 2 & -3 \\ 0 & -2 & 3 & -8 \\ 0 & -2 & 3 & -12 \end{pmatrix} \; -2 \; -2 \qquad \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -6 \end{pmatrix} \; -1 \qquad \begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

Answer: $\underline{\underline{-4}}$

Remember: Each switch of two rows is a $\ominus$

(Not occurring in this example.)