MATH 417 3/17/2023

Next Friday class will be recorded on the course webpage. (Complex numbers.)

no in-person class 3/24

Cramer's rule. System of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \qquad (*)$$

$j=2$

$$\det\begin{pmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & a_{23} & & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & b_n & a_{n3} & & a_{nn} \end{pmatrix} = x_2 \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

(other columns contribute 0).

In the system (*), if $\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn}\end{pmatrix} \neq 0$ then

replace jth column by RHS

the unknowns

$$x_j = \frac{\det\begin{pmatrix} a_{11} & b_1 & a_{1n} \\ \vdots & \vdots & \\ a_{n1} & b_n & a_{nn}\end{pmatrix}}{\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}}$$

(Cramer rule)

Example:

$$\begin{aligned} 5x + 2y + 3z &= 1 \\ 2x + 4y + z &= 2 \\ 3x + 5y + z &= 3 \end{aligned}$$

solution:

$$\det\begin{pmatrix} 5 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 5 & 1\end{pmatrix} = 5\cdot(-1) - 2\cdot(-1) + 3\cdot(-2) = -9$$

$$\boxed{\begin{aligned} x &= 5/9 \\ y &= 4/9 \\ z &= -8/9 \end{aligned}}$$ ✓

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 5 & 1\end{pmatrix} = 1\cdot(-1) - 2\cdot(-1) + 3\cdot(-2) = -5$$

$$\det\begin{pmatrix} 5 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 3 & 1\end{pmatrix} = 5\cdot(-1) - 1\cdot(-1) + 3\cdot 0 = -4$$

$$\det\begin{pmatrix} 5 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 5 & 3\end{pmatrix} = 5\cdot 2 - 2\cdot 0 + 1\cdot(-2) = 8$$

Cramer rule for the inverse matrix: $AX = I$

For jth column of X:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= 1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= 0 \end{aligned}$$

If we replace the kth column of A by the right hand side, we get

$$x_k = \frac{(-1)^{j+k} \det A_{jk}}{\det A}$$

A_{jk} is gotten from A by deleting the jth row and kth column.

same as k

Denoting

$$A^{-1} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix},$$ we have

$$\boxed{x_{ij} = \frac{(-1)^{i+j} \det A_{ji}}{\det A}}$$

Check for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $\quad A^{-1} = \dfrac{1}{a_{11}a_{22}-a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$.

<u>Example</u>: Find A^{-1} where $A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}$ using the Cramer rule.

<u>Solution</u>:

$2\cdot5-1\cdot1 \qquad -(4\cdot5-1\cdot2) \qquad 4\cdot1-2\cdot2$

$$A^{-1} = \frac{1}{9} \begin{pmatrix} 9 & -18 & 0 \\ -3 & 11 & -1 \\ -3 & 5 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & -2 & 0 \\ -1/3 & 11/9 & -1/9 \\ -1/3 & 5/9 & 2/9 \end{pmatrix}}}$$

11 entry of AA^{-1}: $3\cdot9+4\cdot(-3)+2\cdot(-3)$

$-(1\cdot5-2\cdot1)$

$3\cdot5-2\cdot2$

$-(3\cdot1-1\cdot2)$

$1\cdot1-2\cdot2$

$-(3\cdot1-2\cdot4)$

$3\cdot2-1\cdot4$

Example: ? A^{-1} where $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & 3 \\ 1 & 5 & 1 \end{pmatrix}$

Solution:

$$A^{-1} = \frac{-1}{33} \begin{pmatrix} -12 & 9 & -3 \\ 1 & 2 & -8 \\ 7 & -19 & 10 \end{pmatrix} \quad \checkmark$$

HW: 5 In today's problems, must use Cramer rule for full credit.

Use Cramer rule to solve

$$\begin{aligned} 3x + 5y + 8z &= 1 \\ 2x + 2y + 3z &= 3 \\ 8x + y + 2z &= 7 \end{aligned}$$

6 Use Cramer rule to find A^{-1} where

$$A = \begin{pmatrix} 10 & 1 & 2 \\ 1 & 4 & 1 \\ 3 & 2 & 5 \end{pmatrix}.$$