MATH 417 3/20/2023

Trying to understand better linear transformations

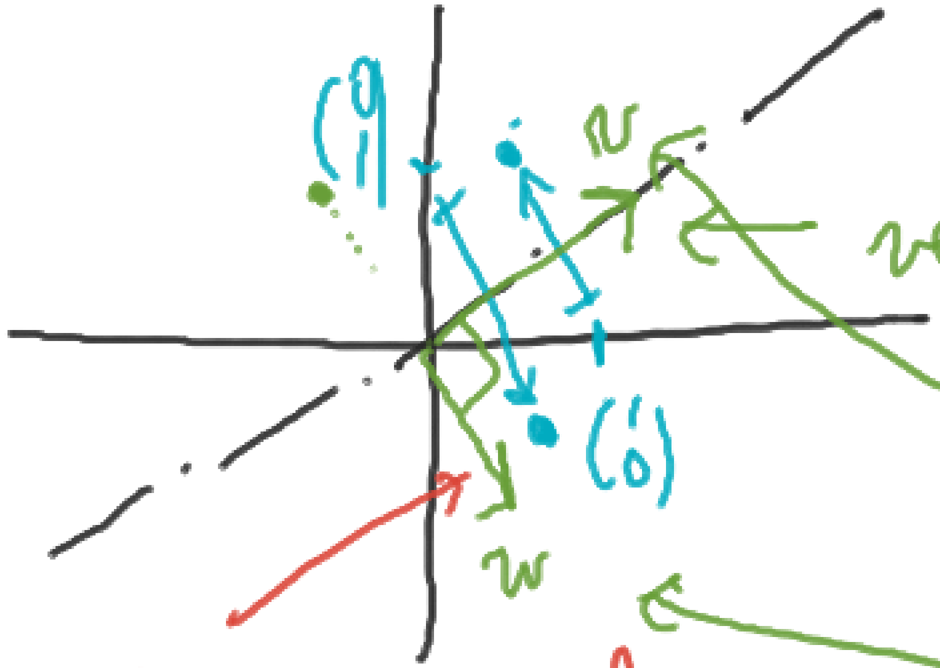
$f: V \to V$.

same domain and codomain.

Example:

$f: \mathbb{R}^2 \to \mathbb{R}^2$

Reflection by an axis



vector v on the axis does not move

$f(v) = v$ (fixed point)

$f(w) = -w$

the perpendicular line to the axis

If we take the basis $B: v, w$ then

same basis in the domain and codomain

${}_B f_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ diagonalization

Example: $f: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(x) = Ax$ where

$${}_E f_E = A = \begin{pmatrix} 2 & 2 \\ 3 & 7 \end{pmatrix}.$$

Can we find non-zero vector v such that

$Av = \lambda v$ for some number λ.

eigenvector (hidden vector, eigenvalue hidden value) ← rarely used today

Solution: $Av = \lambda I v$ $\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(A - \lambda I) v = 0$ $\quad v \neq 0$

$\operatorname{rank}(A - \lambda I) < 2$

$\det(A - \lambda I) = 0$

$(\det(\lambda I - A) = 0)$

$$\det \begin{pmatrix} \lambda - 2 & -2 \\ -3 & \lambda - 7 \end{pmatrix} = 0$$

$$(\lambda - 2)(\lambda - 7) - 6 = \lambda^2 - 9\lambda + 8 = (\lambda - 1)(\lambda - 8)$$

$$\boxed{\text{Eigenvalues: } \lambda = 1, \lambda = 8}$$

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 7 \end{pmatrix} \qquad \lambda = 1, \quad \lambda = 8$$

$\boxed{\lambda = 1}$ $\quad \lambda I - A = \begin{pmatrix} -1 & -2 \\ -3 & -6 \end{pmatrix}$

$(1 \;\; 2)$

$\uparrow$

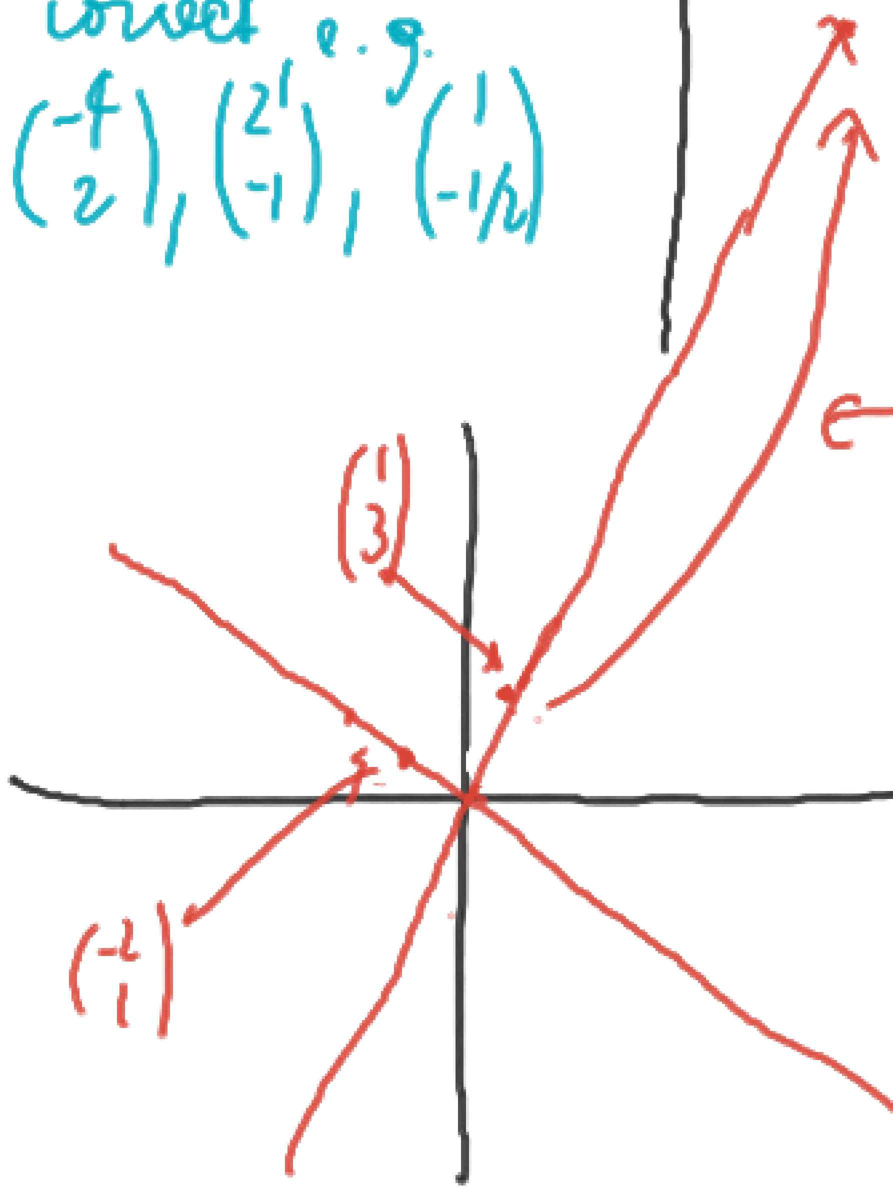
non-pivot

$$\boxed{v = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

any non-zero multiple is also correct e.g. $\begin{pmatrix} -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$

eigenvector of $\lambda = 1$

Geometric picture



$\lambda = 8$

$$\lambda I - A = \begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/3 \end{pmatrix}$$

$\uparrow$

$\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$

$\boxed{v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$ (or any non-zero multiple)

eigenvector for $\lambda = 8$

$Av = 8v$

$f x = \begin{pmatrix} 2 & 2 \\ 3 & 7 \end{pmatrix} x$

$B: \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$${}_B f_B = \begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$$

$C: \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

dominant eigenvalue (largest absolute value)

$${}_C f_C = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}$$

$Av = v$

$f(v) = v$

More comments: $A = \begin{pmatrix}2 & 2\\ 3 & 7\end{pmatrix}$ $\quad B: \begin{pmatrix}1\\ 3\end{pmatrix}, \begin{pmatrix}-2\\ 1\end{pmatrix}$ $\quad D = \begin{pmatrix}8 & 0\\ 0 & 1\end{pmatrix}$

$$B = \begin{pmatrix}1 & -2\\ 3 & 1\end{pmatrix}$$

$$AB = \begin{pmatrix}2 & 2\\ 3 & 7\end{pmatrix}\begin{pmatrix}1 & -2\\ 3 & 1\end{pmatrix} = \rightsquigarrow = \begin{pmatrix}8 & -2\\ 24 & 1\end{pmatrix} = \begin{pmatrix}1 & -2\\ 3 & 1\end{pmatrix}\begin{pmatrix}8 & 0\\ 0 & 1\end{pmatrix}$$

multiply first column by 8

second column by 1

$$= BD$$

We can write diagonalization as finding an invertible matrix B such that

$\boxed{B^{-1}AB = D}$ where D is diagonal.

Example: Find the eigenvectors and eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 2 & 10 \end{pmatrix}$$

Diagonalise the matrix A

Solution: Solve

$$0 = \det(\lambda I - A) = \det\begin{pmatrix} \lambda-2 & -1 & -2 \\ -2 & \lambda-3 & -4 \\ -4 & -2 & \lambda-10 \end{pmatrix}$$

Friday lecture: recorded
3/24
complex numbers
posted today

$$= (\lambda-2)(\lambda-3)(\lambda-10) - 8(\lambda-2) - 2(\lambda-10) - 16 - 8 - 8(\lambda-3)$$

$$= \lambda^3 - 15\lambda^2 + 38\lambda - 24 \quad = (\lambda-1)(\lambda-2)(\lambda-12)$$

$$\begin{array}{r} \lambda^2 - 14\lambda + 24 \\ \lambda - 1 \overline{)\lambda^3 - 15\lambda^2 + 38\lambda - 24} \\ -\lambda^3 + \lambda^2 \\ \hline -14\lambda^2 + 38\lambda - 24 \\ 14\lambda^2 - 14\lambda \\ \hline 24\lambda - 24 \end{array}$$

$(\lambda-2)(\lambda-12)$

Eigenvalues: $\lambda = 1$

$\lambda = 2$

$\lambda = 12$

eigenvectors next time

HW ① Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}.$$

② Diagonalize the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}.$$