MATH 417 3/22/2023

Diagonalizing the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 2 & 10 \end{pmatrix}$.

We found $\lambda = 1, 2, 12$

$\boxed{\lambda = 12}$

$$(\lambda I - A) = \begin{pmatrix} 10 & -1 & -2 \\ -2 & 9 & -4 \\ -4 & -2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -9 & 4 \\ 10 & -1 & -2 \\ -4 & -2 & 2 \end{pmatrix} \begin{matrix} -5 \\ 2 \end{matrix} \quad \begin{pmatrix} 2 & -9 & 4 \\ 0 & 44 & -22 \\ \cancel{0} & \cancel{-20} & \cancel{10} \end{pmatrix} \tfrac{1}{44} \quad \begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \end{pmatrix} \quad \text{eigenvector: } \underline{\underline{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}}$$

$$\begin{pmatrix} 2 & -9 & 4 \\ 0 & 1 & -1/2 \end{pmatrix} 9 \quad \begin{pmatrix} 2 & 0 & -1/2 \\ 0 & 1 & -1/2 \end{pmatrix} \tfrac{1}{2}$$

$\uparrow$

4

$$\begin{pmatrix} s/4 \\ s/2 \\ s \end{pmatrix}$$

$\boxed{\lambda = 2}$

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 2 & 10 \end{pmatrix}$$

$$\lambda I - A = \begin{pmatrix} 0 & -1 & -2 \\ -2 & -1 & -4 \\ \cancel{-4} & \cancel{-2} & \cancel{-8} \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \end{pmatrix} \; -1$$

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \; 2 \; {}^1\!/\!_2$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \boxed{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}} \text{ eigenvector}$$

$s = 1$

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 2 & 10 \end{pmatrix} \qquad \boxed{\lambda = 1}$$

$$\lambda I - A = \begin{pmatrix} -1 & -1 & -2 \\ \cancel{-2} & \cancel{-2} & \cancel{-4} \\ -4 & -2 & -9 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2 \\ -4 & -2 & -9 \end{pmatrix} 4$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \end{pmatrix} 2\ 1/2$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1/2 \end{pmatrix} -1 \qquad \begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \end{pmatrix}$$

$$\uparrow$$
$$2$$

$$\boxed{\begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}}$$

Diagonalising the matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 2 & 10 \end{pmatrix}$

Eigenvalues: $\lambda = 12, 2, 1$

Corresponding eigenvectors: $\underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix}}_{B}$

$$AB = B \begin{pmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & -1 & -5 \\ 2 & -2 & 1 \\ 4 & 1 & 2 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & -1 & -5 \\ 2 & -2 & 1 \\ 4 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

no need to figure the inverse matrix here.

Note: eigenvectors from different eigenvalues are necessarily linearly independent.

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

eigenvectors of λ_1 λ_2 λ_k different

Multiply by A: $a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \cdots + a_k \lambda_k v_k = 0$

can eliminate v_1. | Eventually, if there is only $k=1$, it would say that $av=0 \therefore v=0$ which is impossible. (contradiction)

Example: diagonalise $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$

Solution:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda-3 & -1 & -1 \\ 1 & \lambda-1 & 1 \\ -2 & -2 & \lambda-4 \end{pmatrix} = (\lambda-3)(\lambda-1)(\lambda-4) + 2 + 2$$

$$-2(\lambda-1) + 2(\lambda-3) + (\lambda-4)$$

$$= \lambda^3 - 8\lambda^2 + 20\lambda - 16$$

if the coeff. $=1$

other coeffs are integers then rational roots are integers dividing the constant term.

$\lambda^3 - 8\lambda^2 + 20\lambda - 16$

$\lambda = 2$ $\quad 8 \cdot 32 + 40 - 16 = 0$

$$
\begin{array}{r}
\lambda^2 - 6\lambda + 8 \\
\lambda - 2 \,\big)\, \overline{\lambda^3 - 8\lambda^2 + 20\lambda - 16} \\
\underline{-\lambda^3 + 2\lambda^2 \qquad\qquad\quad} \\
-6\lambda^2 + 20\lambda - 16 \\
\underline{6\lambda^2 - 12\lambda \qquad} \\
8\lambda - 16 \\
\underline{\qquad} \\
0
\end{array}
$$

$\lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$

$\lambda^3 - 8\lambda^2 + 20\lambda - 16$

$= (\lambda - 2)(\lambda - 2)(\lambda - 4)$

$ax^2 + bx + c = 0$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Eigenvalues: 2, 2, 4

degenerate eigenvalue

$\lambda = 2$

has algebraic multiplicity 2.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix} \qquad \boxed{\lambda = 4}$$

$$\lambda I - A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -2 & -2 & 0 \end{pmatrix} \begin{matrix} -1 \\ 2 \end{matrix}$$

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & 2 \\ \cancel{0} & \cancel{-4} & \cancel{-2} \end{pmatrix} \, 1/4 \qquad \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \end{pmatrix} \, 1 \qquad \begin{array}{c} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{pmatrix} \\ \qquad\qquad\quad 2 \end{array} \qquad \boxed{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix} \qquad \lambda = 2$$

$$\lambda I - A = \begin{pmatrix} \cancel{-1} & \cancel{-1} & \cancel{-1} \\ 1 & 1 & 1 \\ \cancel{-2} & \cancel{-2} & \cancel{-2} \end{pmatrix}$$

The solution space of the matrix $\lambda I - A$ is called the <u>eigenspace</u> of λ.

We need to find a basis of the eigenspace.

RREF

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} & \uparrow & \uparrow \\ & s & t \end{matrix}$$

$$\boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

diagonalisering $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 1 & -1 \\ 2 & 2 & 4 \end{pmatrix}$:

eigenwaarde | alg. multipl. uit 2

$$\underbrace{\overset{4}{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}} \qquad \overset{2}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}}_{B}$$

$$\boxed{\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

HW

3 Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

4 Diagonalize the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & 6 \\ 1 & 2 & 0 \end{pmatrix}$$

Friday class recorded

Complex numbers