MATH 417 3/24/2023

no real solution to $x^2+1=0$

Introduce a solution $x = i$.

$$i^2 = -1.$$

$$(4+6i)+(3+7i) = 7+13i$$

$$(4+6i)(3+7i) = 12+14i+18i-42 = -30+32i$$

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

$$\underbrace{(c+di)}_{z}\,\underbrace{(c-di)}_{\bar{z}\ \longleftarrow\ \text{complex conjugato.}} = c^2+d^2$$

$$\frac{5+3i}{4+3i} = \frac{(5+3i)(4-3i)}{(4+3i)(4-3i)} = \frac{29-3i}{25} =$$

$$= \frac{29}{25} - \frac{3}{25} i .$$

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i$$

Complex numbers form a <u>field</u> $\longleftarrow +, -, \cdot$ have the usual properties (like real numbers)

$\rightsquigarrow$ also divide (except by 0).

We can study vector spaces over a field

Field of complex numbers: $\mathbb{C}$

Fundamental theorem of algebra:

Every non-zero polynomial over $\mathbb{C}$ has a root: $x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$ always has a solution.

Geometric interpretation of complex numbers:

$$\mathbb{C} \cong \mathbb{R}^2$$

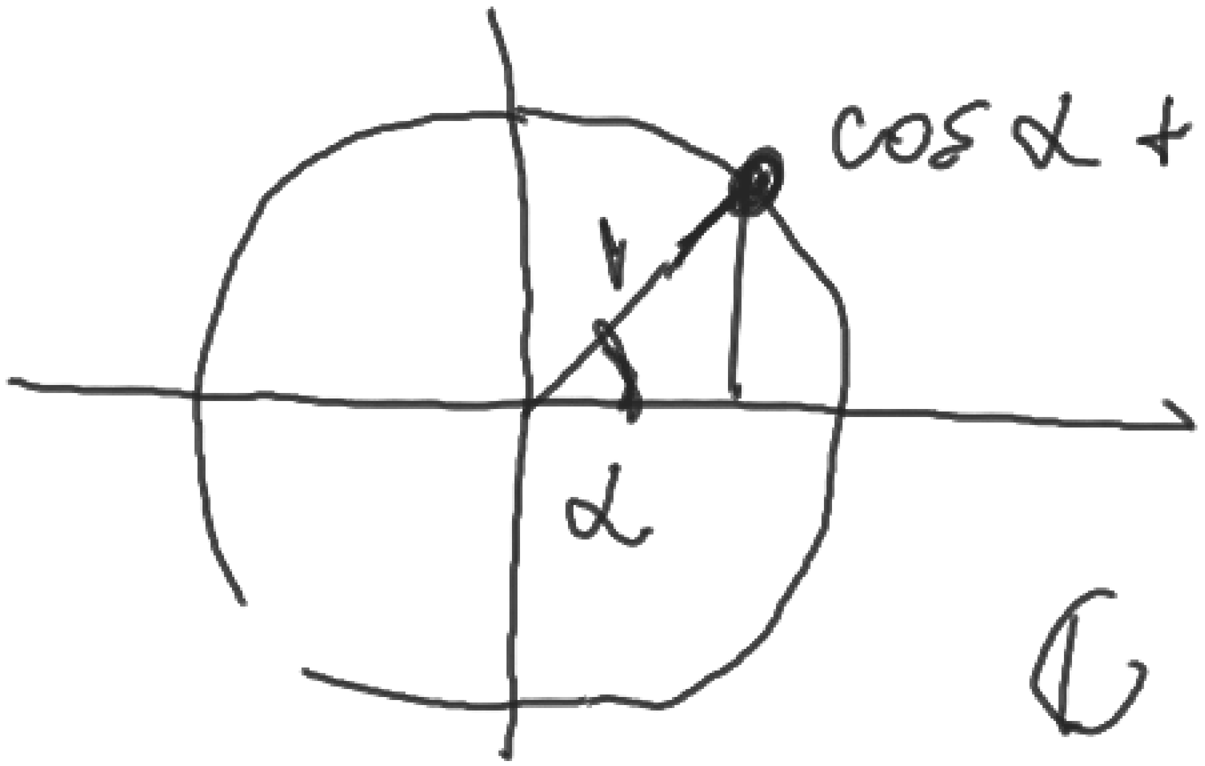
$$a+bi \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

$a+bi = z$

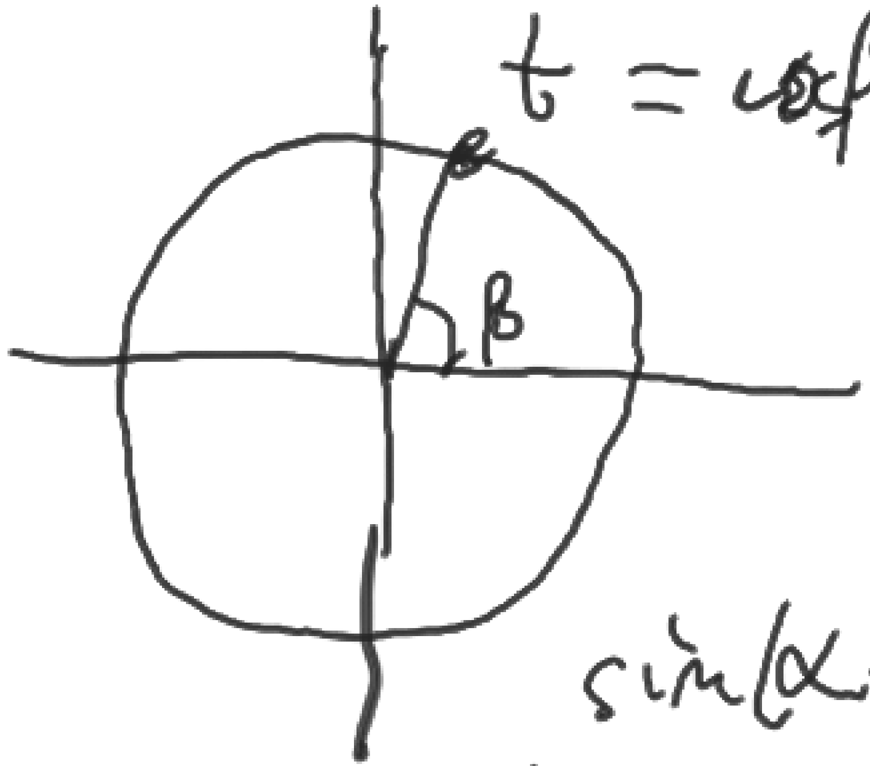
$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$$

$$|3+2i| = \sqrt{3^2+2^2} = \sqrt{13}$$

$$\boxed{|zt| = |z|\,|t|}$$



$\cos\alpha + i\sin\alpha = z$

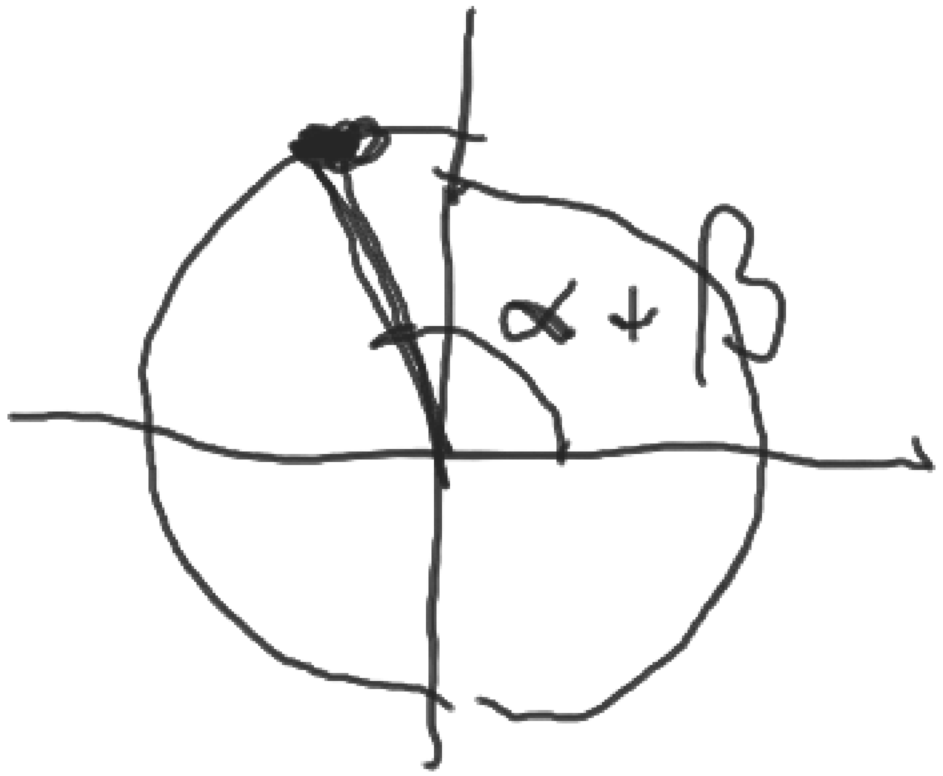


$t = \cos\beta + i\sin\beta$

$\mathbb{C}$

$$zt = \overbrace{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}^{\cos(\alpha+\beta)} + i\overbrace{(\cos\alpha\sin\beta + \sin\alpha\cos\beta)}^{\sin(\alpha+\beta)}$$

zt



$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \cdots$$

$$e^{x+y} = e^x e^y$$

Plugging in complex numbers:

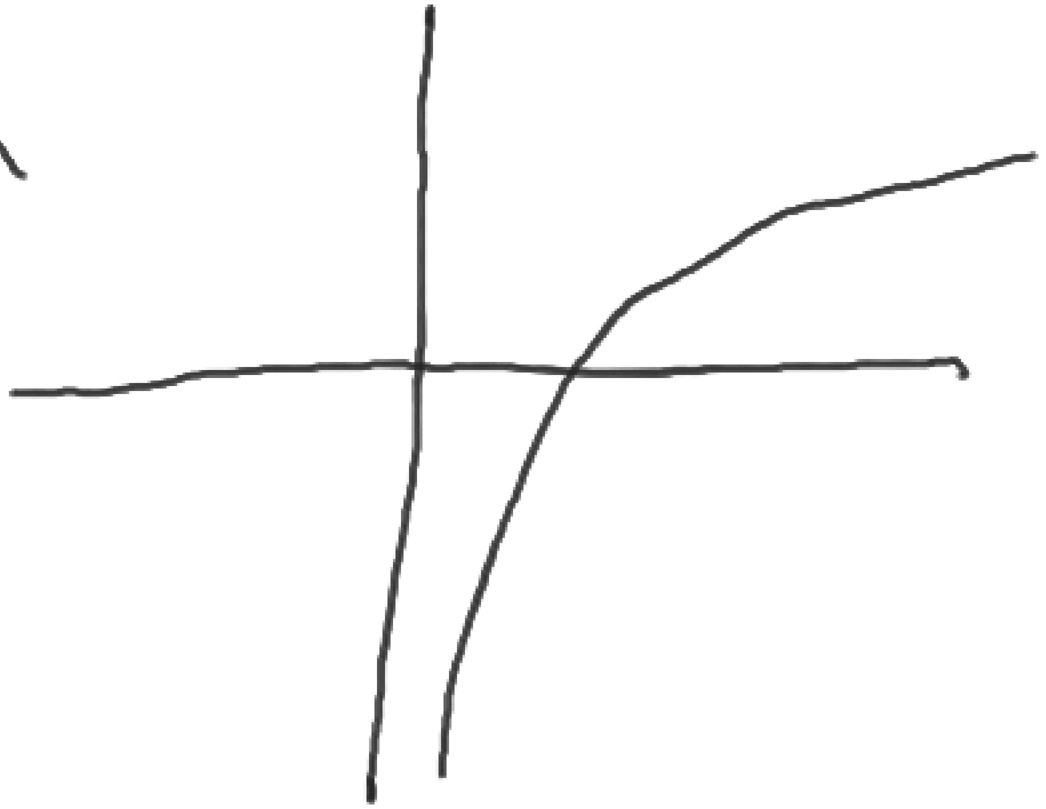
$$\boxed{e^{ix} = \cos x + i \sin x}$$ Euler formula.

$$e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}.$$

$$\boxed{e^{2\pi i} = 1}$$

What about $\ln x$ for $x \in \mathbb{C}$?

$x \in \mathbb{R}$



$\ln x = y$

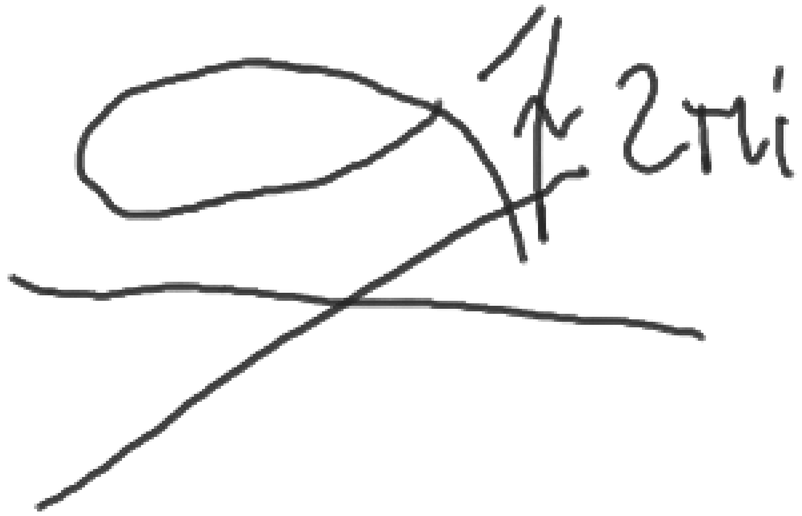
$x \in \mathbb{C}$

$e^{\ln x} = x$

no solution if $x = 0$

defined only up to adding $2\pi k i$, $k \in \mathbb{Z}$

$$e^{2+\pi i/6} = e^2 \frac{\sqrt{3}}{2} + \frac{e^2}{2} i$$



The complex logarithm is a "multivalued function" which increases by $2\pi i$ when we go around the unit circle once counter-clockwise.

$$\ln(z) = \ln|z| + (\beta + 2\pi k)\, i$$

Proving the fundamental theorem of algebra:

$$p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

If $a_0 = 0$, $z=0$ is a root. Assume $a_0 \neq 0$.

$$\frac{p(rz)}{r^n} = z^n + \frac{a_{n-1} z^{n-1}}{r} + \cdots + \frac{a_0}{r^n} \sim z^n$$

$r \in [0, \infty)$ If z is on the unit circle $r >> 0$

$\frac{p(rz)}{r^n}$ (hence also $p(rz)$) will go around the origin n times counterclockwise when $r >> 0$. $\therefore$ $\ln(p(rz))$ increases by $2\pi i n$.

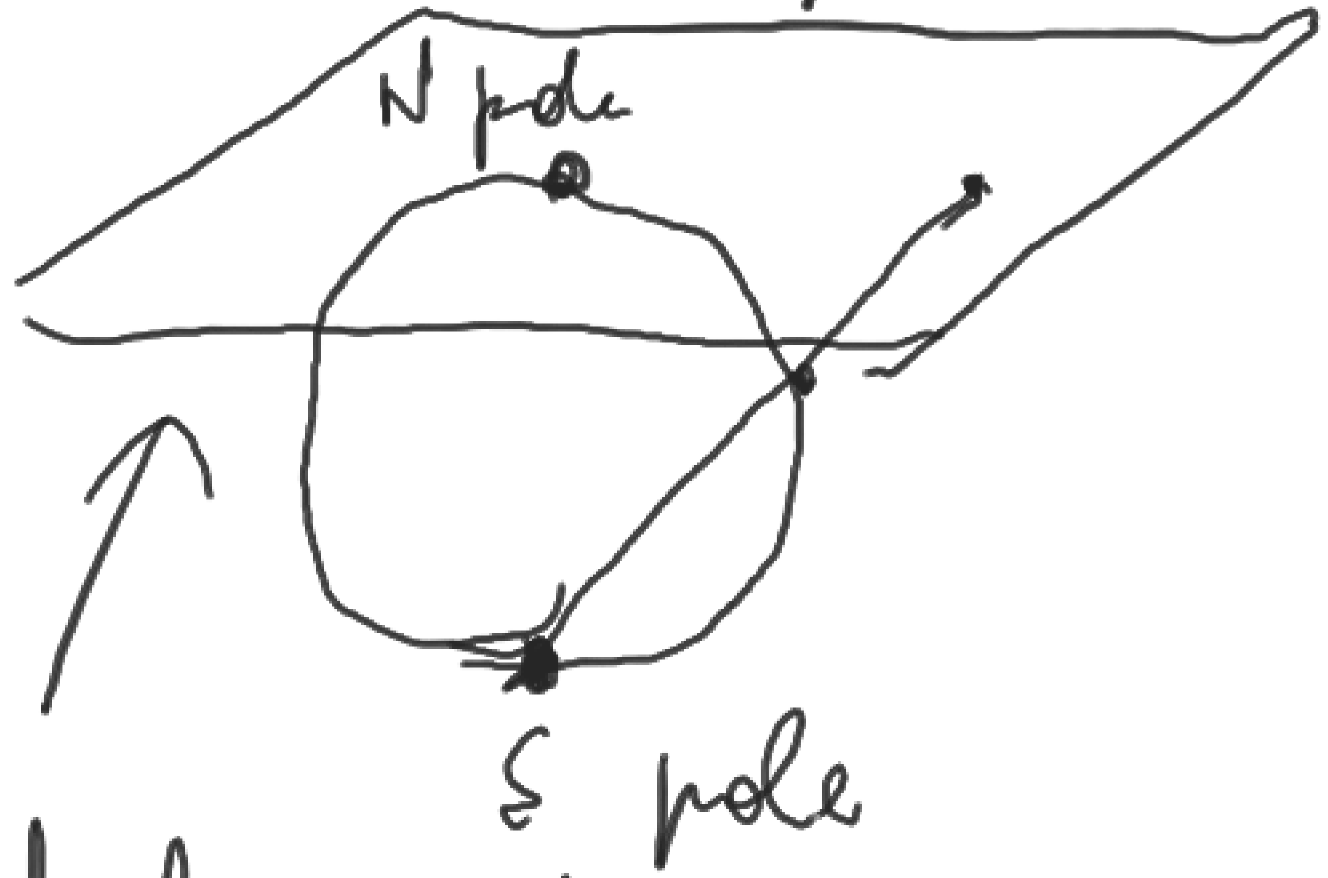
(z goes around the unit circle once counterclockwise)

But not for $r = 0$!

So for some $r \in (0, \infty)$, some z on the unit circle, $\ln(p(rz))$ must be undefined

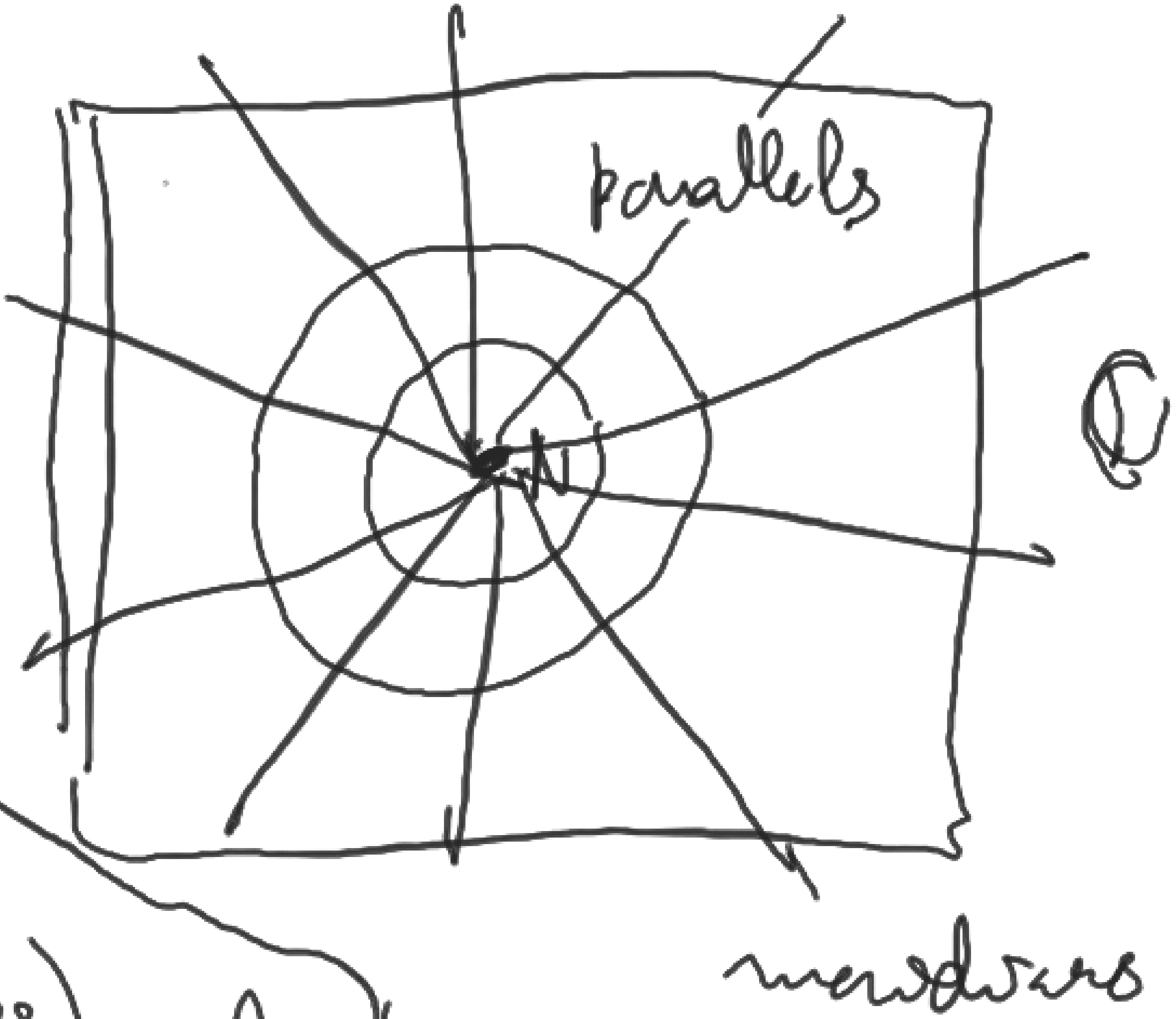
$\therefore p(rz) = 0$. $\square$

Mercator projection

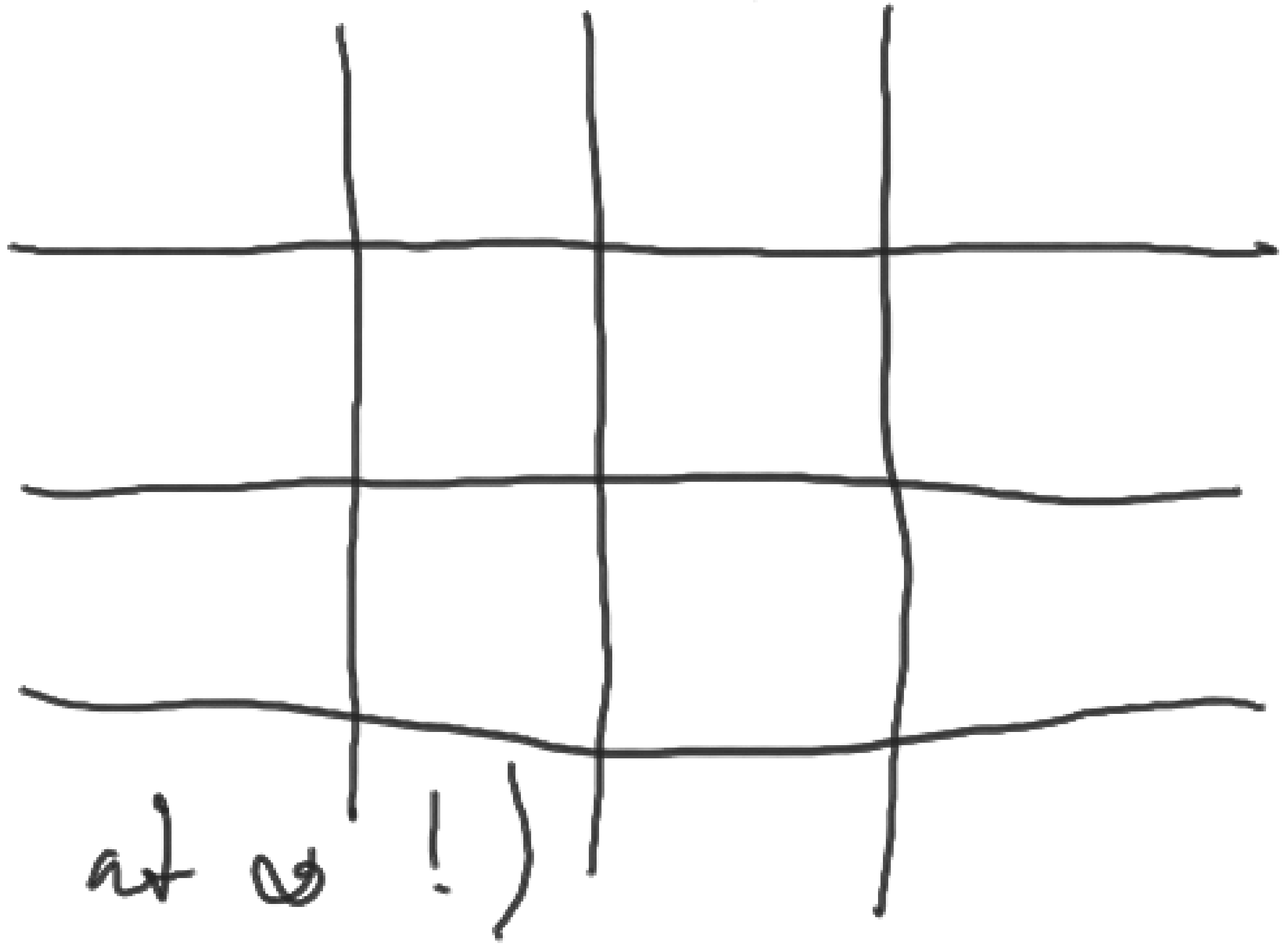


radial projection (preserves angles)

$\ln z$



The Mercator projection is the complex logarithm of the polar radial projection.



Meridians, parallels, rumb lines (constant azimuth) are [illegible] lines

(The poles are at ∞ !)

HW 5

a) $(4+7i)(2-6i)$

b) $\dfrac{4+5i}{3+5i}$

c) $(1+i)^6$

d) $e^{3+\pi i/3}$

e) $\ln(1-i)$