MATH 417 3/27/2023

When we diagonalize a matrix A, we come up with an invertible matrix B such that

$$B^{-1}AB = D \text{ is diagonal.} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

More generally, we call two matrices A_1, A_2 <u>similar</u> when there exists an invertible matrix B such that

$$B^{-1}A_1B = A_2$$

Similarity is an equivalence relation.

$A_1 \sim_s A_2$ then $A_2 \sim_s A_1$ $(BA_2B^{-1} = A)$

$A \sim_s A$ Transitivity: $A_1 \sim_s A_2$ & $A_2 \sim_s A_3 \Rightarrow A_1 \sim_s A_3$.

If two matrices A_1, A_2 are diagonalisable and have the same eigenvalues with the same multiplicities, then they are similar.

Example: Are the matrices

$$A_1 = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

similar?

Solution: $\det(\lambda I - A_1) = \det\begin{pmatrix} \lambda-2 & -2 \\ -1 & \lambda-3 \end{pmatrix} = (\lambda-2)(\lambda-3)-2 = \lambda^2-5\lambda+4$
$= (\lambda-1)(\lambda-4)$

$A_1 \sim \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ $\quad A_2 \sim \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ NOT SIMILAR

$\det(\lambda I - A_2) = \det\begin{pmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{pmatrix} = (\lambda-2)(\lambda-2)-1 = \lambda^2-4\lambda+3 =$
$= (\lambda-1)(\lambda-3)$

Follow up: Are the matrices

$$A_1 = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

similar?

$$A_1 \sim_s \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\det(\lambda I - A_3) = \det \begin{pmatrix} \lambda - 2 & -1 \\ -2 & \lambda - 3 \end{pmatrix} =$$

$$= (\lambda - 2)(\lambda - 3) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$$

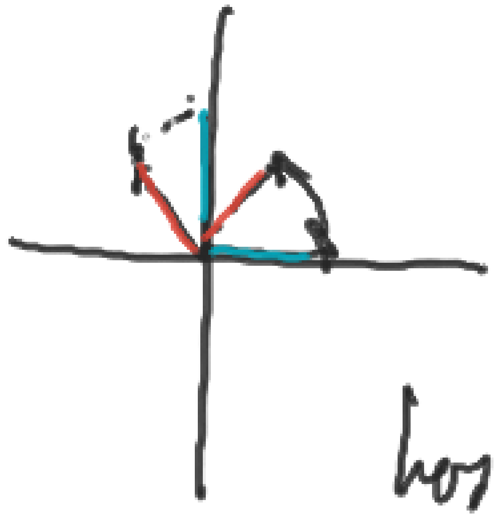
$$A_3 \sim_s \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$\boxed{\text{yes, } A_1 \sim_s A_3}$ similar.

Some important twists to the question of similarity:

① Complex numbers.

The simplest example of a linear transformation that has complex eigenvalues is a rotation:



Rotation by a non-trivial angle (not $2\pi k$, $k \in \mathbb{Z}$) has no real eigenvectors.

$$ax^2 + bx + c = 0$$

has a solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

discriminant

If $b^2 - 4ac < 0$ the solution is complex (use the rule $\sqrt{-1} = \pm i$)

What is happening?

Example: rotation by $\frac{\pi}{2}$ counterclockwise.

90°

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Finding eigenvalues:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$$

Eigenvalues: $\lambda = i$ $\quad\quad$ $\lambda = -i$

complex conjugate

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(complex)

Eigenvectors for $\lambda = i$:

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \quad 2 \cdot i \qquad \text{RREF} \quad \begin{pmatrix} 1 & -i \end{pmatrix}$$

$$\begin{pmatrix} 1 & -i \\ \cancel{-1} & \cancel{i} \end{pmatrix}$$

Eigenvector: $\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\overline{a+bi} = a-bi$

$\lambda = -i$ (complex conjugate)

eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Follow-up: diagonalize the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ over $\mathbb{C}$

the complex numbers

$$\boxed{\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}$$

Example: Are the matrices

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

similar?

Solution: A_2 has eigenvalues $i, -i$.

$\det(\lambda I - A_1) = \det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$

A_1 eigenvalues: $i, -i$.

Complex eigenvalues can be used to check similarity

$\boxed{\text{Yes, they are similar}}$

Even though the eigenvalues are complex, if the matrices are similar, they are similar over the real numbers.

Example: Find the complex eigenvalues and eigenvectors of the matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Solution:

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 1 \end{pmatrix} = (\lambda-1)(\lambda-1) + 1 =$$

$$= \lambda^2 - 2\lambda + 2 = (\lambda - (1+i))(\lambda - (1-i))$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$

Eigenvectors: $\lambda = 1 + i$

$$\lambda I - A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \quad 2 \ -i$$

$$\begin{pmatrix} 1 & -i \\ \cancel{-1} & \cancel{i} \end{pmatrix} \qquad \begin{pmatrix} i \\ 1 \end{pmatrix} \leftarrow \text{eigenvector} \rightarrow \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$\lambda = 1 - i$

H/W: 1. Are the matrices

$$A_1 = \begin{pmatrix} 3 & 1 \\ 4 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

similar?

2. Diagonalise over the complex numbers:

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}.$$

Two diagonal matrices with the same eigenvalues in different order (same multiplicities) are similar

$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \sim_s \begin{pmatrix} 2 & & \\ & 1 & \\ & & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & 2 \end{pmatrix} \not\sim_s \begin{pmatrix} 2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$