MATH 417 3/29/2023

<u>Example</u>: Is the matrix

$$A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

diagonalizable?

<u>Solution</u>:

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-3 & -1 \\ 4 & \lambda+1 \end{pmatrix} =$$

$$= (\lambda-3)(\lambda+1)+4 = \lambda^2-2\lambda+1 = (\lambda-1)^2$$

One eigenvalue $\lambda = 1$ with multiplicity 2.

$A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ Only eigenvalue $\lambda = 1$ <u>algebraic multiplicity</u> $= 2$

If A were diagonalisable, we would have $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$B^{-1}AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underbrace{B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B^{-1}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = A$$

false!

A is <u>not</u> diagonalisable.

Find eigenvectors:

$$(1 \cdot I - A) = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 \end{pmatrix}$$

Eigenvector: $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (or any multiple)

<u>Geometric multiplicity</u> $= \dim(\text{Eigenspace}) = 1$.

$A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ eigenvalue $\lambda = 1$ (algebraic multiplicity $= 2$)

eigenvector: $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (geometric multiplicity $= 1$).

$f : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(v) = (\lambda I - A)v = (I - A)v = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} v.$

$\lambda = 1$

$Im(f) = \{ f(v) \mid v \in \mathbb{R}^2 \} = V$ $\quad \boxed{f : V \to V = 0}$

$f : V \longrightarrow V$ $\quad$ In this example, $V = \left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle$ (because 2×2-matrix)

A commutes with itself and the unit matrix.

$\dim \operatorname{Ker}(\lambda I - A) = 1$

$\uparrow$

solution space

$\downarrow$

$\dim \operatorname{Ker}((\lambda I - A)^2) = 2$

$\lambda = 1 \qquad A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \qquad \lambda I - A = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$

$$(\lambda I - A)^2 = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

eigenvector $v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

We can look for a vector w which satisfies

$$(\lambda I - A) w = -v \qquad w = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} +1 \\ -2 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -2 & -1 & +1 \\ \cancel{4} & \cancel{2} & \cancel{-2} \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1/2 & -1/2 \end{array}\right)$$

$\uparrow$ parametric: say, 0

e.g. $w = \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -1/2 \\ 0 \end{pmatrix} = \begin{pmatrix} +1 \\ -2 \end{pmatrix} = -v$$

$$A = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$$

Jordan block

B : v, w

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$$

$$A \underbrace{\begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix}}_{B} = \begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix} \boxed{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

$$Aw = w + v$$

$$(\lambda I - A) w = -v$$

$\lambda = 1$

check:

$$\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -3/2 \\ 2 & 2 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -3/2 \\ 2 & 2 \end{pmatrix}$$

Putting the matrix A in Jordan form:

$$\underbrace{\begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix}^{-1}}_{B^{-1}} \underbrace{\begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} -1 & -1/2 \\ 2 & 0 \end{pmatrix}}_{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

In general, a <u>Jordan block</u> of size k with eigenvalue λ is:

$$J_k(\lambda) = k\left\{ \underbrace{\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}}_{k} \right.$$

(λ's on the diagonal
1's right above the diagonal
0 elsewhere).

$$J_1(\lambda) = (\lambda) \qquad 1\times 1 \text{ matrix}$$

A general Jordan form consists of Jordan blocks along the diagonal:

Jordan blocks

$$\begin{pmatrix} \boxed{\begin{matrix}3&1&0\\0&3&1\\0&0&3\end{matrix}} & & & 0 \\ & \boxed{i} & & \\ & & \boxed{\begin{matrix}2&1\\0&2\end{matrix}} & \\ 0 & & & \boxed{\begin{matrix}1&1&0&0\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{matrix}} \end{pmatrix}$$

Every complex matrix A has a Jordan form

($B^{-1}AB$ is in Jordan form)

The Jordan form is uniquely defined up to the order of Jordan blocks.

__Example:__ Put the matrix

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

into Jordan form.

__Solution:__ $\det(\lambda I - A) = \det\begin{pmatrix} \lambda-3 & -1 \\ 1 & \lambda-1 \end{pmatrix} = (\lambda-3)(\lambda-1)+1 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$

$Aw = 2w + v$

$\lambda = 2$ algebraic multiplicity 2

$(2I - A)w = -v$

$(2I - A) = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ eigenvector $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\sim (1 \ \ 1)$

$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & -1 \end{array}\right) \quad w = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$\boxed{\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}}$$

$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix}$

$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 2 & 1 \end{pmatrix}$ ✓

HW: 3. Put the matrix

$$A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$$

in Jordan form.

4. Write a matrix in Jordan form which has

2 Jordan blocks of size 3 with eigenvalue -1

1 Jordan block of size 2 with eigenvalue $1+i$

1 Jordan block of size 1 with eigenvalue -1

1 Jordan block of size 1 with eigenvalue 5.