MATH 417 3/31/2023

Finding the Jordan form of a matrix A.

Step 1: Find the eigenvalues.

Step 2: For an eigenvalue λ, find the dimensions of the solution spaces of $(\lambda I - A)^k$, $k = (0), 1, 2, \ldots$

$\boxed{\dim(\mathrm{Ker}((\lambda I - A)^k))}$

Example: $\lambda = 3$

$$A = \begin{pmatrix} \boxed{\begin{matrix}3 & 1\\ & 3\end{matrix}} & & & \\ & \boxed{3} & & \\ & & \boxed{\begin{matrix}3 & 1\\ & 3\end{matrix}} & \\ & & & \boxed{\begin{matrix}3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1\\ & & & 3\end{matrix}} \end{pmatrix}$$

What are the dimensions

$\dim \operatorname{Ker} (3I - A)^k$, $k = (0), 1, 2, \dots$?

Solution:

k	0	1	2	3	4	5	
$\dim \operatorname{Ker}(3I-A)^k$	0	4	7	8	9	9	.....

the number $\dim \operatorname{Ker}(3I-A)^2$ + the number of blocks of size ≥ 3

$\dim \operatorname{Ker}(3I-A)^3$ + # of blocks of size ≥ 4

algebraic multiplicity of the eigenvalue $\lambda = 3$

the number of blocks (= geometric multiplicity)

the number $\dim \operatorname{Ker}(3I-A)^1$ + the number of blocks of size ≥ 2

0 4 7 8 9 9

differences 4 3 1 1 0

# of blocks 1 2 0 1

# of blocks ≥ 2

# of blocks ≥ 3

# of blocks ≥ 4

exact numbers of blocks of a given size

Example: Let A be a matrix with eigenvalues $2, -4$.

always 0

k	0	1	2	3	4
$\dim \operatorname{Ker}(2I-A)^k$	0	4	6	7	7
$\dim \operatorname{Ker}(4I-A)^k$	0	3	4	4	4

Find the Jordan form of A.

start with 0

include one repeating number

Solution: $\lambda = 2$

$$\begin{array}{ccccccccc} 0 & & 4 & & 6 & & 7 & & 7 \\ & 4 & & 2 & & 1 & & 0 & \\ & & 2 & & 1 & & 1 & & \end{array}$$

numbers of blocks of sizes 1,2,3

$\lambda = -4$

$$\begin{array}{ccccccc} 0 & & 3 & & 4 & & 4 \\ & 3 & & 1 & & 0 & \\ & & 2 & & 1 & & \end{array}$$

$$\begin{pmatrix}
2 & & & & & & & & & & \\
 & 2 & & & & & & & & & \\
 & & 2 & 1 & & & & & & & \\
 & & & 2 & & & & & & & \\
 & & & & 2 & 1 & & & & & \\
 & & & & & 2 & 1 & & & & \\
 & & & & & & 2 & & & & \\
 & & & & & & & -4 & & & \\
 & & & & & & & & -4 & & \\
 & & & & & & & & & -4 & 1 \\
 & & & & & & & & & & -4
\end{pmatrix}$$

Linear differential equations

$u' = A\,u$ $\qquad u = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ unknown functions (independent variable $= t$)

square matrix of constants

Example: Solve $u' = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} u$.

$u = \begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad x' = 3x + y$, $\quad y' = 2x + 4y$

Solution:
Suppose we diagonalize the matrix.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-3 & -1 \\ -2 & \lambda-4 \end{pmatrix} = (\lambda-3)(\lambda-4) - 2 = \lambda^2 - 7\lambda + 10 = (\lambda-2)(\lambda-5)$$

$\boxed{\lambda = 2}$ $\quad 2I - A = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}$ eigenvector: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\boxed{\lambda = 5}$ $\quad 5I - A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}$ eigenvector: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$$

Substitute: $u = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} v$ $\qquad v = \begin{pmatrix} r \\ s \end{pmatrix}$

$v' = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} v$ $\qquad r' = 2r$, $\; r = Ke^{2t}$ $\qquad s' = 5s$, $\; s = Le^{5t}$

$$u = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} Ke^{2t} \\ Le^{5t} \end{pmatrix} = \begin{pmatrix} -Ke^{2t} + Le^{5t} \\ Ke^{2t} + 2Le^{5t} \end{pmatrix}$$

$$u' = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} u \qquad u = \begin{pmatrix} -Ke^{2t} + Le^{5t} \\ Ke^{2t} + 2Le^{5t} \end{pmatrix}$$

$$u' = \begin{pmatrix} -2Ke^{2t} + 5Le^{5t} \\ 2Ke^{2t} + 10Le^{5t} \end{pmatrix} \qquad \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -Ke^{2t} + Le^{5t} \\ Ke^{2t} + 2Le^{5t} \end{pmatrix} = \quad \checkmark$$

$$= \begin{pmatrix} -3Ke^{2t} + 3Le^{5t} + Ke^{2t} + 2Le^{5t} \\ -2Ke^{2t} + 2Le^{5t} + 4Ke^{2t} + 8Le^{5t} \end{pmatrix}$$

Example: Solve the system of differential equations:

$$u' = \underbrace{\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}}_{A} u .$$

Solution:

Diagonalise.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-2 & -2 \\ -2 & \lambda-5 \end{pmatrix} = (\lambda-2)(\lambda-5)-4$$

$$= \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6)$$

$\boxed{\lambda = 1}$

$$\lambda I - A = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$\boxed{\lambda = 6}$

$$\lambda I - A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$$

Substitute $u = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} v \qquad v = \begin{pmatrix} r \\ s \end{pmatrix}$

$$r' = 1 \cdot r \qquad s' = 6 \cdot s$$

$$r = Ke^{t} \qquad s = Le^{6t}$$

$$\boxed{u = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} Ke^{t} \\ Le^{6t} \end{pmatrix} = \begin{pmatrix} -2Ke^{t} + Le^{6t} \\ Ke^{t} + 2Le^{6t} \end{pmatrix}} \quad ✓$$

$$u' = \begin{pmatrix} -2Ke^{t} + 6Le^{6t} \\ Ke^{t} + 12Le^{6t} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} -2Ke^{t} + Le^{6t} \\ Ke^{t} + 2Le^{6t} \end{pmatrix} = \begin{pmatrix} -4Ke^{t} + 2Le^{6t} + 2Ke^{t} + 4Le^{6t} \\ -4Ke^{t} + 2Le^{6t} + 5Ke^{t} + 10Le^{6t} \end{pmatrix}$$

HW (5) A matrix A has eigenvalues $\lambda = 2, 5$. We have:

k	1	2	3	4	
$\dim \ker (2I - A)^k$	3	5	6	6	...
$\dim \ker (5I - A)^k$	4	5	5	...	

Find the Jordan form of A.

(6) Solve the system of linear differential equations

$$u' = \begin{pmatrix} 4 & 1 \\ 4 & 4 \end{pmatrix} u.$$