MATH 417 04/03/2023

What I said last time on differential equations also applies to complex eigenvalues (just take, say, the real part).

Example: Solve the system of linear differential equations

(a) $u' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u$.

(b) $u' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} u$.

partially decoupled system.

Solution: $u = \begin{pmatrix} x \\ y \end{pmatrix}$ (independent var. t)

(a) $x' = y$
$y' = 0$.

y is constant

$$\boxed{\begin{aligned} x &= at + b \\ y &= a \end{aligned}}$$

(b) $\quad u' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} u \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x + y \\ -2y \end{pmatrix}$

$$x = e^{-2t}(at + b)$$

$$x' = \underbrace{-2e^{-2t}(at+b)}_{-2x} + \underbrace{e^{-2t}a}_{y}$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{-2t}(at+b) \\ e^{-2t}a \end{pmatrix}}$$

$n=3$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = J_3(\lambda)$$

general Jordan block:

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \underbrace{\begin{pmatrix} +\lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & +\lambda \end{pmatrix}}_{J_n(\lambda)} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$x_1 = e^{\lambda t}\underbrace{(a_{n-1}t^{n-1} + \dots + a_0)}_{\text{polynomial of degree } \leq n-1}$$

$x_2 = \dots$

(Real) symmetrical matrices.

Example: diagonalise

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}.$$

Solution:

$$0 = \det(\lambda I - A) = \det\begin{pmatrix} \lambda-6 & -2 \\ -2 & \lambda-3 \end{pmatrix} = (\lambda-6)(\lambda-3) - 4 = \lambda^2 - 9\lambda + 14$$

$$= (\lambda-2)(\lambda-7)$$

$\boxed{\lambda=2}$ $\begin{pmatrix} 4 & -2 \\ -2 & -1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\boxed{\lambda=7}$ $\begin{pmatrix} +1 & -2 \\ -2 & 4 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\boxed{\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0$$

If A is a real symmetric matrix, then

$$A = A^T. \quad \text{(by definition)}$$

Suppose $\lambda \neq \mu$ are eigenvalues. Eigenvector $\lambda \cdots u$

$\mu \cdots v$

$$Au = \lambda u \qquad \underbrace{Av = \mu v.}$$

transpose: $v^T A^T = \mu v^T$

$$\mu v^T u = v^T A u = v^T \lambda u = \lambda v^T u$$

$$v \cdot u = v^T u = 0$$

orthogonal = perpendicular

$\therefore$ If A is a real symmetric matrix and u, v are eigenvectors of different eigenvalues then $u \cdot v = 0$. $(u \perp v)$

$$\underset{\text{has orthogonal columns.}}{B^{-1}}\,\overset{\text{symmetric}}{A}\,B = \overset{\text{diagonal}}{D}$$

We can also make the columns of B to have length 1:

$$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

real vector

In our example, $A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$

For B, we used $\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

$$\left\| \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\| = \sqrt{2^2+1^2} = \sqrt{5}$$

$$\left\| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\| = \sqrt{1^2+2^2} = \sqrt{5}$$

We can replace B by $\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$

The columns are orthogonal and have length 1.

an orthogonal matrix

We call a matrix B <u>orthogonal</u> if its columns are orthogonal and their lengths are 1.

<u>Algebraically</u>, let $B = \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix}$. columns

$\underbrace{u_i \cdot u_i}_{u_i^T u_i} = 1$

$\underbrace{u_i \cdot u_j}_{u_i^T u_j} = 0$ when $i \neq j$.

$$B^T B = \begin{pmatrix} u_1^T \\ \vdots \\ u_n^T \end{pmatrix} \begin{pmatrix} u_1 & \dots & u_n \end{pmatrix} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = I$$

rows

$\boxed{B \text{ Being an orthogonal matrix is equivalent to the condition } B^T B = I}$

$B^{-1} = B^T$

Example: $B = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$ $\quad B^T = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix} = B^{-1}$.

$\det B = -1$

reflection by a line through the origin

We could have also chosen

$$B = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \quad \text{Then } B^T = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = B^{-1}.$$

$\det B = 1.$

rotation around the origin.

In general, when B is an orthogonal matrix, then

$$(\det B)^2 = \det B^T \det B = \det B^T B = 1$$

$\therefore$ $\det B = 1$ or $\det B = -1$.

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}$$

Geometrically: If B is an orthogonal matrix then the linear transformation $f(v) = Bv$ takes the standard basis $(e_1, \ldots e_n) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ into a basis of vectors which are orthogonal and have length 1.

Orthonormal basis

The linear transformation f corresponding to an orthogonal matrix B preserves lengths and angles. If additionally $\det B = 1$, these are the transformation you can use to move a mechanical device without deforming it.

HW 1 : Solve the system of linear differential equations

$$u' = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} u.$$

2 : Let $A = \begin{pmatrix} 4 & 3 \\ 3 & 12 \end{pmatrix}$. Find an <u>orthogonal</u> matrix B such that $B^T A B$ is diagonal.