MATH 417 4/5/2023

Recall that an orthogonal matrix is a real $n \times n$ matrix B such that

$$B^T B = I.$$

This is equivalent to the condition that the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(x) = Bx$ preserves distances and angles.

The matrix product $B_1 B_2$ of two orthogonal matrices of the same size $n \times n$ is again orthogonal. If B is orthogonal, then B^{-1} is orthogonal.

This is expressed by saying that orthogonal $n \times n$ matrices form a group with respect to the operation of matrix multiplication.

the operation is associative, $(B_1 B_2) B_3 = B_1 (B_2 B_3)$
we have a unit element $IB = BI = B$
and also inverse B^{-1}.

The group of orthogonal $n \times n$-matrices is called the orthogonal group $O(n)$.
The determinant of an orthogonal matrix is 1 or -1. The orthogonal matrices with determinant 1 are called special orthogonal. They form the special orthogonal group $SO(n)$.

The group $SO(3)$ is the group of rigid (= distance preserving) linear transformations. What do the elements $B \in SO(3)$ look like?

The eigenvalues of B can be complex, but have to have absolute value 1.

(also complex conjugate must be present) if not real eigenvalue

There must be a real eigenvalue (an odd-degree real polynomial always has a real root ). So B must have an eigenvalue 1.

Every $\underbrace{3\times 3 \text{ special orthogonal matrix}}_{B \in SO(3)}$ is the matrix of a rotation (w.r.t. the standard basis).

Our interest in orthogonal matrices is mostly in orthogonal diagonalisation of (real) symmetric matrices.

$A^T = A$

If A is a real symmetric matrix then all the eigenvalues of A are real and there exists an orthogonal matrix B such that

$$\underbrace{B^T A B}_{= B^{-1} A B} = D \quad \text{is diagonal.}$$

The spectral theorem, The principal axes theorem

Why are the eigenvalues real? $Av = \lambda v \quad v \neq 0. \qquad A\bar{v} = \bar{\lambda}\bar{v}$

complex conjugations

A symmetrical: $\bar{v}^T A = \bar{v}^T A^T = \bar{\lambda}\bar{v}^T.$ $\quad \bar{\lambda}\bar{v}^T v = \bar{v}^T A v = \bar{v}^T \lambda v = \lambda \bar{v}^T v$

So we have $\bar{\lambda}\bar{v}^T v = \lambda \bar{v}^T v$. If $v \neq 0$

If $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is a complex vector, $\bar{v}^T v = \underbrace{\bar{x}_1 x_1}_{\|x_1\|^2} + \cdots + \underbrace{\bar{x}_n x_n}_{\|x_n\|^2} > 0$.

$\bar{v}^T = (\bar{x}_1 \cdots \bar{x}_n)$

$\therefore \bar{\lambda} = \lambda$. Thus, the eigenvalue λ is real.

The complex dot-product: If $u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, the

$\in \mathbb{C}$

dot-product $u \cdot v = \bar{u}^T v = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$.

Then $\|u\|^2 = u \cdot u$ also for a complex vector u.

Example: Let $u = \begin{pmatrix} 2+i \\ 1-i \end{pmatrix}$, $v = \begin{pmatrix} 1+i \\ 3+i \end{pmatrix}$. Compute $\|u\|$, $\|v\|$, $u \cdot v$

Solution: $\|u\| = \sqrt{\underbrace{(2-i)(2+i)}_{2^2+1^2} + \underbrace{(1+i)(1-i)}_{1^2+1^2}} = \sqrt{7}$

$\|v\| = \sqrt{\underbrace{(1-i)(1+i)}_{1^2+1^2} + \underbrace{(3-i)(3+i)}_{3^2+1^2}} = \sqrt{12}$

$\overline{2+i}$ $\qquad$ $\overline{1-i}$

$$u \cdot v = (2-i)(1+i) + (1+i)(3+i)$$
$$= 2+1+2i-i+3-1+3i+i$$
$$= \underline{\underline{5+5i}}.$$

We proved the spectral theorem for the case of non-degenerate eigenvalues. What about degenerate eigenvalues?

Example: Orthogonally diagonalise:

$$A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

Solution:

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-5 & -2 & -2 \\ -2 & \lambda-2 & -1 \\ -2 & -1 & \lambda-2 \end{pmatrix} = (\lambda-5)(\lambda-2)(\lambda-2) - 4 - 4$$

$$- 4(\lambda-2) - 4(\lambda-2) - (\lambda-5) = \lambda^3 - 9\lambda^2 + 15\lambda - 7$$

$$\begin{array}{r|l} & \lambda^2 - 8\lambda + 7 \\ \hline \lambda - 1 & \lambda^3 - 9\lambda^2 + 15\lambda - 7 \\ & -\lambda^3 + \lambda^2 \\ \hline & -8\lambda^2 + 15\lambda - 7 \\ & 8\lambda^2 - 8\lambda \\ \hline & 7\lambda - 7 \end{array}$$

$(\lambda-1)(\lambda-7)$

$\boxed{\lambda = 1}$ double

$\boxed{\lambda = 7}$

$A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

$\underline{\lambda = 1}$ $\begin{pmatrix} -4 & -2 & -2 \\ -2 & -1 & -1 \\ -2 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 1/2 \end{pmatrix}$

$\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ degenerate eigenvalue, <u>not orthogonal</u>

exact calculation requires an <u>orthogonalisation process</u>

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(to be continued)

HW 3 $u = \begin{pmatrix} 4+i \\ 2-i \end{pmatrix}$, $v = \begin{pmatrix} 3-i \\ 1+i \end{pmatrix}$. Calculate $\|u\|$, $\|v\|$

$u \cdot v \; (= \bar{u}^T v)$.

4 Find the eigenvalues of

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$ and their multiplicities.