MATH 417 4/7/2023

Continuing the example: Orthogonally diagonalise

can do this because A is symmetric

$$A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

So far: double eigenvalue $\lambda = 1$ eigenvectors: $\underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}_{\text{not orthogonal! (degenerate eigenvalue)}}$

Third eigenvalue: $\lambda = 7$

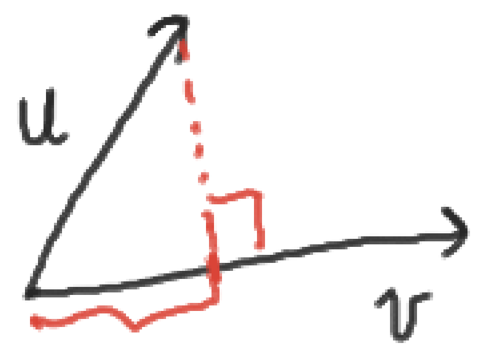
We could solve: $\ker \begin{pmatrix} +2 & -2 & -2 \\ -2 & 5 & -1 \\ -2 & -1 & 5 \end{pmatrix} \rightsquigarrow \ldots$

orthogonal to $\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Orthogonalization process (Gram-Schmidt)

$$\underset{u}{\begin{pmatrix}-1\\2\\0\end{pmatrix}} \quad \underset{v}{\begin{pmatrix}-1\\0\\2\end{pmatrix}}$$



$$\mathrm{proj}_v(u) = \boxed{\frac{u\cdot v}{v\cdot v}}\, v$$

a multiple of v

number

If $\|v\|=1$, $(u\cdot v)\,v$

In general overcounting by factor $\underbrace{\|v\|\;\|v\|}_{v\cdot v}$

$$(u-\mathrm{proj}_v(u))\cdot v = 0$$

$$u-\mathrm{proj}_v(u) = \begin{pmatrix}-1\\2\\0\end{pmatrix} - \frac{1}{5}\begin{pmatrix}-1\\0\\2\end{pmatrix} =$$

5 · the expression

$$\begin{pmatrix}-5\\10\\0\end{pmatrix} - \begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}-4\\10\\-2\end{pmatrix} \overset{1/2}{\sim} \begin{pmatrix}-2\\5\\-1\end{pmatrix}$$

$$\begin{pmatrix}-2\\5\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\2\end{pmatrix} = 0 \quad ✓$$

$$\underbrace{\begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ -2/\sqrt{30} & 5/\sqrt{30} & -1/\sqrt{30} \end{pmatrix}}_{B^T} \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \underbrace{\begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \\ 1/\sqrt{6} & 2/\sqrt{5} & -1/\sqrt{30} \end{pmatrix}}_{B} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

eigenvectors of 1 : $\frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ $\frac{1}{\sqrt{30}}\begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}$

eigenvectors of 7 : $\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$B = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{5} & -2/\sqrt{30} \\ 1/\sqrt{6} & 0 & 5/\sqrt{30} \\ 1/\sqrt{6} & 2/\sqrt{5} & -1/\sqrt{30} \end{pmatrix}$$

orthogonal matrix

Follow-up question: Is B a rotation?

? $\det B \overset{?}{=} 1$

$\det B = -1$ $\underline{\underline{\text{NO}}}$

$$\det \begin{pmatrix} 2 & -1 & -2 \\ 1 & 0 & 5 \\ 1 & 2 & -1 \end{pmatrix} = -4 - 5 - 1 - 20 = -30 < 0$$

To fix it, change sign of the first column:

$$\begin{pmatrix} -2/\sqrt{6} & -1/\sqrt{5} & -2/\sqrt{30} \\ -1/\sqrt{6} & 0 & 5/\sqrt{30} \\ -1/\sqrt{6} & 2/\sqrt{5} & -1/\sqrt{30} \end{pmatrix}$$

Gram-Schmidt process for row vectors leads to the <u>orthogonal row echelon form</u> OREF

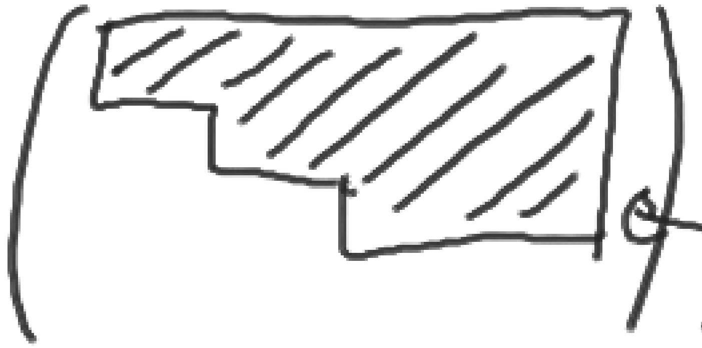
↑

① A row echelon form where ② the pivots > 0

③ rows are orthonormal

(orthogonal, $\underbrace{\text{have length 1}}_{\text{do that in the end!}}$).

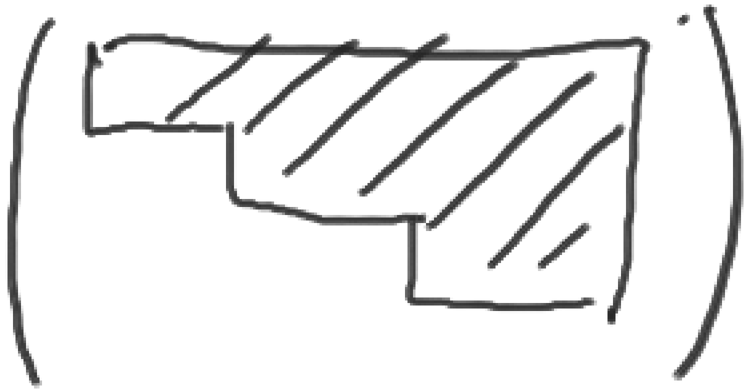
First find a row echelon form



① keep the last row w

② then orthonormalize the previous row v with respect to w.

$v - \frac{v \cdot w}{w \cdot w} w$ | orthogonalize each row with respect to the rows below it.



v
$w_1 \leftarrow$
$w_2 \leftarrow$ if these are orthogonal then replace v by

$$v - \frac{v\cdot w_1}{w_1\cdot w_1}\, w_1 - \frac{v\cdot w_2}{w_2\cdot w_2}\, w_2$$

$$\vdots$$

Example: Find the orthogonal row echelon form of

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

Answer: $\begin{pmatrix} 3/\sqrt{51} & 4/\sqrt{51} & 5/\sqrt{51} & 1/\sqrt{51} \\ 0 & 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$

Solution:

-1

$\|v\| = \sqrt{3}$

$\|\ \| = \sqrt{9+16+25+1} = \sqrt{51}$

$$\begin{array}{c} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow u \\ \leftarrow v \end{array}$$

$$u \rightsquigarrow u - \frac{u\cdot v}{v\cdot v}\, v \sim (v\cdot v)u - (u\cdot v)v = 3(1\ 1\ 2\ 0) - (-1)(0\ 1\ -1\ 1) = (3\ 4\ 5\ 1)$$

HW 5 Orthogonally diagonalise:

$$A = \begin{pmatrix} 10 & 3 & 3 \\ 3 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

6 Find the orthogonal row echelon form of the matrix

$$\begin{pmatrix} 2 & 1 & -1 & 0 \\ 3 & 1 & 1 & 1 \end{pmatrix}$$

<u>Note</u>: The real symmetric matrices which most often occur in statistics have <u>positive eigenvalues</u>

Such matrices A are called <u>positive definite</u> and have the property that for every column vector v (of the size = # of rows of A)

$$v^T A v > 0$$

Orthogonal diagonalisation:

$$B^T A B = D \leftarrow \text{diagonal with positive eigenvalues}$$

$$\underbrace{u^T B^T}_{(Bu)^T} A \underbrace{Bu}_{\text{any } v.} = u^T D u > 0$$

$$= \lambda_1 x_1^2 + \dots + \lambda_n x_n^2$$

$$u = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$