MATH 417 4/10/2023

Exam Monday 4/17 in class

Review Friday 4/14

Last HW due Friday 4/14 10 AM

(assigned today and Wednesday)

Example: Consider the special orthogonal matrix

$$A = \begin{pmatrix} 9/25 & -4/5 & -12/25 \\ 12/25 & 3/5 & -16/25 \\ 4/5 & 0 & 3/5 \end{pmatrix}$$

$$-\underbrace{9 \cdot 12}_{} - \underbrace{12 \cdot 16}_{} + 12 \cdot 25 = 0$$

$$-108 \quad -192 \quad +300 \quad = 0 \checkmark$$

Find the axis and angle of rotation.

solution:

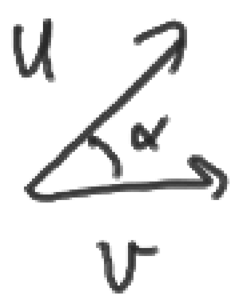
$$(I - A) = \begin{pmatrix} 16/25 & 4/5 & 12/25 \\ -12/5 & 2/5 & 16/25 \\ -4/5 & 0 & 2/5 \end{pmatrix}$$

eigenvector:
$$\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\boxed{\text{axis of rotation} \quad \left\{ \begin{pmatrix} t \\ -2t \\ 2t \end{pmatrix} \middle| t \in \mathbb{R} \right\}}$$

$$\sim \begin{pmatrix} 4 & 5 & 3 \\ -6 & 5 & 8 \\ -2 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ -3 \end{matrix} \; 2 \qquad \begin{pmatrix} \cancel{0 \;\; 5 \;\; 5} \\ 0 & 5 & 5 \\ -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \end{pmatrix}$$

Find the angle:



$$\frac{u\cdot v}{\|u\|\cdot\|v\|} = \cos\alpha$$

Choose any vector orthogonal to the eigenvector $\begin{pmatrix}1\\-2\\2\end{pmatrix}$:

Say, $\begin{pmatrix}0\\1\\1\end{pmatrix} =: u$

$$Au = \begin{pmatrix} 9/25 & -4/5 & -12/25 \\ 12/25 & 3/5 & -16/25 \\ 4/5 & 0 & 3/5 \end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}-32/25\\-1/25\\3/5\end{pmatrix}$$

Taking a multiple won't change the angle.

$$v = \begin{pmatrix}-32\\-1\\15\end{pmatrix}$$

$$\frac{u\cdot v}{\|u\|\|v\|} = \frac{14}{\sqrt{2}\sqrt{1750}} = \cos\alpha$$

$$\begin{array}{r} 1024 \\ +\quad 1 \\ +\ 625 \\ \hline 1750 \end{array}$$

$$\boxed{\alpha = \arccos\frac{14}{\sqrt{3500}}}$$

Statistical analysis using the spectral theorem.

Random variable: measurements from the researcher's point of view which produce some values with some $\underbrace{\text{probability}}_{\text{how often a given value occurs?}}$

could be AI

$0 \leq p \leq 1$

never

always

A random vector:

a collection of n random variables that you measure on the same sample.

e.g. a photograph, each pixel $\rightsquigarrow$ numerical value.

Can we recognize a photograph?

We measure statistics of $\underbrace{\text{random variables.}}$

say, each random variable can attain only finitely many num. values.

<u>discrete random variable</u>

Most basic information: <u>expectation</u> (average)

$$E(X) = \sum_{x \text{ values}} x \cdot P(X = x).$$

<u>Example</u>: What is the expectation of casting a die?

$$\frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4$$

$$+ \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = \frac{21}{6} = \underline{\underline{\frac{7}{2}}}$$

Variance: $\mathrm{var}(X) = E(X^2) - E(X)^2$

$$= E\left(\left(X - E(X)\right)^2\right)$$

For two random variables X, Y:

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= E\left(\left(X - E(X)\right)\left(Y - E(Y)\right)\right)$$

$$\mathrm{var}(X) = \mathrm{cov}(X,X).$$

Statistical version of a dot product

The die example

$$E(X^2) = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6}$$

$$= \frac{91}{6}$$

$$\mathrm{var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 =$$

$$\frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} =$$

$$= \boxed{\frac{35}{12}}$$

Suppose we have a random vector $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$. We can form the <u>covariance matrix</u>

var(X_1) ...

$$\operatorname{cov}(X) = \begin{pmatrix} \operatorname{cov}(X_1,X_1) & \operatorname{cov}(X_1,X_2) & \cdots & \operatorname{cov}(X_1,X_n) \\ \operatorname{cov}(X_2,X_1) & \operatorname{cov}(X_2,X_2) & & \operatorname{cov}(X_2,X_n) \\ & & & \\ \operatorname{cov}(X_n,X_1) & \operatorname{cov}(X_n,X_2) & & \operatorname{cov}(X_n,X_n) \end{pmatrix}$$

The covariance matrix of the random vector X.

A real symmetric matrix

Suppose we find an <u>orthogonal matrix B</u> such that

$$B^T \operatorname{cov}(X)\, B = D \text{ is } \underline{\text{diagonal}}.$$

$\leftarrow$ positive definite matrix

$$D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix} \qquad \lambda_1 > \lambda_2 > \lambda_3 > \cdots$$

$$\boxed{\operatorname{cov}(B^T X) = D}$$

$$B^T X = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \end{pmatrix}$$

Y_1: the principal traits in this sample space.

dominant trait

Only the first few dominant traits are important in recognizing the sample (the remaining eigenvalues are insignificant)

~ 1990s <100 to recognize a face.

(HW) Due Friday 4/14 10 AM

(1) Find the axis and angle of rotation of the special orthogonal matrix

$$A = \begin{pmatrix} 4/9 & -4/9 & -7/9 \\ 1/9 & 8/9 & -4/9 \\ 8/9 & 1/9 & 4/9 \end{pmatrix}$$

(2) Suppose X is the random variable of choosing a number 1, 2, 3, 4, 5 (every number is equally likely). Find $E(X)$, $\mathrm{var}(X)$.