MATH 417 4/12/2023

Test 3: - Cramer rule
- eigenvalues and eigenvectors, including the complex case, diagonalizing a matrix
- Jordan form
- Real symmetric matrices, orthogonal diagonalization (including the case of a degenerate eigenvalue - orthogonalization process) Gram-Schmidt
- Orthogonal row echelon form.

Example: A simple personality test gives scores 0, 1, 2 on two different questions. Three people took the test and scored

		person 1	person 2	person 3
X	Q1	2	0	1
Y	Q2	1	0	2

Based on this, identify the principal traits of the population and the dominant trait.

Solution: $E(X) = \frac{2+0+1}{3} = 1$ $\qquad E(Y) = \frac{1+0+2}{3} = 1$

$E(X^2) = \frac{4+0+1}{3} = \frac{5}{3}$ $\qquad E(Y^2) = \frac{1+0+4}{3} = \frac{5}{3}$

$\mathrm{var}(X) = \frac{5}{3} - \underbrace{1}_{E(X)^2} = \frac{2}{3}$ $\qquad \mathrm{var}(Y) = \frac{5}{3} - \underbrace{1}_{E(Y)^2} = \frac{2}{3}$

$E(XY) = \frac{2\cdot1 + 0\cdot0 + 1\cdot2}{3} = \frac{4}{3}$ $\qquad \mathrm{cov}(X,Y) = \frac{4}{3} - \underbrace{1}_{E(X)E(Y)} = 1/3$

The $\operatorname{cov}\begin{pmatrix}X\\Y\end{pmatrix} = \begin{pmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{pmatrix}$

linearly scaling the whole matrix by a number is ok.

$$A = \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

The principal traits are

$$B^T\begin{pmatrix}X\\Y\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix} = \begin{pmatrix}X/\sqrt{2} + Y/\sqrt{2}\\ -X/\sqrt{2} + Y/\sqrt{2}\end{pmatrix}$$

Answer:

Principal traits: $X/\sqrt{2} + Y/\sqrt{2}$, $\leftarrow$ dominant

$-X/\sqrt{2} + Y/\sqrt{2}$

$$\det(\lambda I - A) = \begin{pmatrix}\lambda-2 & -1\\ -1 & \lambda-2\end{pmatrix} =$$

$$= (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda - 3 = (\lambda-1)(\lambda-3).$$

Eigenvalues: $\lambda = 3$, $\lambda = 1$

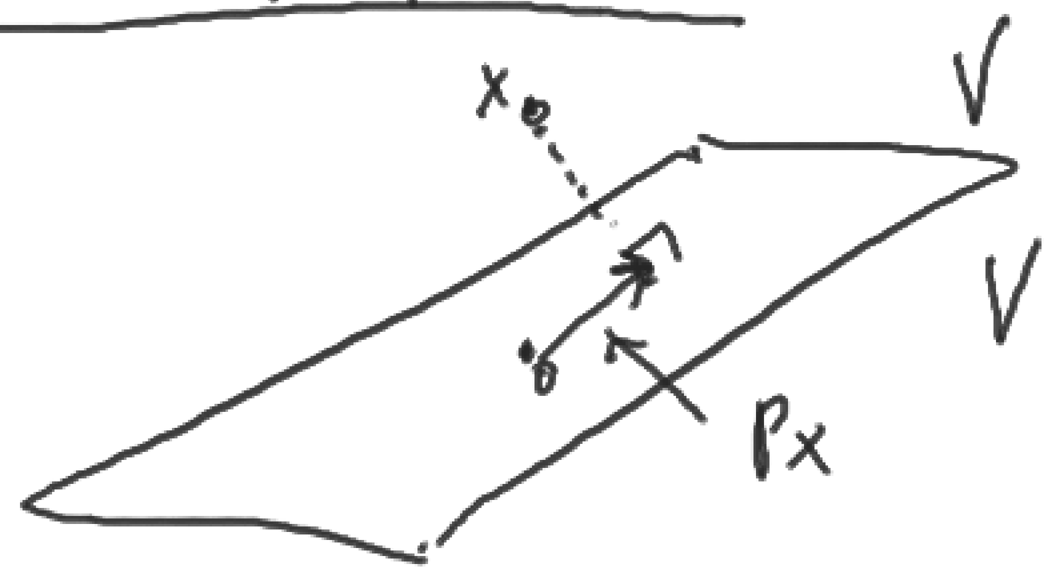
$$\begin{pmatrix}3-2 & -1\\ -1 & 3-2\end{pmatrix} = \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \qquad \begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}$$

eigenvector: $\begin{pmatrix}1\\1\end{pmatrix}$ $\left\|\begin{pmatrix}1\\1\end{pmatrix}\right\| = \sqrt{2}$ eigenvector: $\begin{pmatrix}-1\\1\end{pmatrix}$ $\left\|\begin{pmatrix}-1\\1\end{pmatrix}\right\| = \sqrt{2}$

$$\underbrace{\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}}_{B^T}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\underbrace{\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}}_{B} = \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$$

associated with the dominant eigenvalue.

Orthogonal projection



$V \subseteq \mathbb{R}^n$

$x - Px \perp x \quad ((x - Px) \cdot x = 0$

$Px \in V.$

V = column space of a matrix A $= \{Ay \mid y \in \mathbb{R}^k\}$

$$\text{n rows} \left\{ \underbrace{\begin{pmatrix} \; A \; \end{pmatrix}}_{k \text{ columns}} \right.$$

we assume rank $A = k$

$$(x - Ay) \cdot Az = 0$$

supposed to hold for every $z \in \mathbb{R}^k$

$$\cancel{z^T} A^T (x - Ay) = 0 \quad \text{(matrix multiplication)}$$

$$A^T x - A^T A y = 0$$

$A^T A$ = $k \times k$ matrix symmetrical called the Gram matrix

$$(A^T A)^{-1} A^T x = y$$

rank $A^T A$ = rank A.

If $= k$ (as we assumed), then the Gram matrix is invertible.

The projection formula:

$$\boxed{Px = A(A^T A)^{-1} A^T x}$$

Example: Find the matrix of the orthogonal projection in $\mathbb{R}^3$ onto the plane

$$\left\{ s\begin{pmatrix}1\\2\\1\end{pmatrix} + t\begin{pmatrix}1\\1\\0\end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}.$$

Solution: $A = \begin{pmatrix}1 & 1\\ 2 & 1\\ 1 & 0\end{pmatrix}$ Gram matrix: $A^TA = \begin{pmatrix}1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}6 & 3\\ 3 & 2\end{pmatrix}$

$$(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix}2 & -3\\ -3 & 6\end{pmatrix}$$

Answer:

$$A(A^TA)^{-1}A^T = \frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & -3\\ -3 & 6\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 1 & 1 & 0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1 & 1 & 2\\ 3 & 0 & -3\end{pmatrix} = \boxed{\frac{1}{3}\begin{pmatrix}2 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 2\end{pmatrix}}$$

Recall if A is an $n \times k$ matrix of rank k, the Gram matrix is the $k \times k$ matrix $A^T A$.

If $k = n$
$\det(A^T A) = \det(A)^2$

$\sqrt{\det(A^T A)}$ is the quantity replacing the determinant for non-square matrices. For example, it gives the k-volume of the parallelepiped in $\mathbb{R}^n$ spanned by the columns of A.

Example: Calculate the area of the parallelogram in $\mathbb{R}^4$ spanned by the vectors $\begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\2\end{pmatrix}$.

$$\left\{ s\begin{pmatrix}1\\2\\1\\1\end{pmatrix} + t\begin{pmatrix}1\\0\\1\\2\end{pmatrix} \,\middle|\, 0 \leq s,t \leq 1 \right\}$$

Solution: $A = \begin{pmatrix}1 & 1\\2 & 0\\1 & 1\\1 & 2\end{pmatrix}$ $\quad A^TA = \begin{pmatrix}1 & 2 & 1 & 1\\1 & 0 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 0\\1 & 1\\1 & 2\end{pmatrix} = \begin{pmatrix}7 & 4\\4 & 6\end{pmatrix}$

Answer: $\sqrt{\det\begin{pmatrix}7 & 4\\4 & 6\end{pmatrix}} = \underline{\underline{\sqrt{26}}}$.

$42 - 16 = 26$

HW 3 In a two question personality test, 7 people answered

Q1	1	1	1	2	0	2	0
Q2	0	2	0	2	1	1	1

If we denote by X, Y the random variables corresponding to the questions Q1, Q2, find the principal traits of the population and the dominant trait.

4. Calculate the matrix of the orthogonal projection in $\mathbb{R}^4$ onto the plane spanned by $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

5. Find the area of the parallelogram in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$.