MATH 417

4/14/2023

Review: Cramer Rule

$$10x + 9y = 15$$
$$4x + 2y = 9$$

$$\boxed{x = \frac{-51}{-16} = \frac{51}{16}, \quad y = \frac{30}{-16} = -\frac{30}{16}}$$

$$\det\begin{pmatrix} 10 & 9 \\ 4 & 2 \end{pmatrix} = -16$$

$$\det\begin{pmatrix} 15 & 9 \\ 9 & 2 \end{pmatrix} = -51$$

$$\det\begin{pmatrix} 10 & 15 \\ 4 & 9 \end{pmatrix} = 30$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 4 & 1 & 2 \end{pmatrix}^{-1} = \boxed{\frac{1}{3} \begin{pmatrix} 5 & -3 & -1 \\ 2 & 0 & -1 \\ -11 & 6 & 4 \end{pmatrix}}$$

ji

$$A^{-1} = \frac{1}{\det A}_{i \to j} \left((-1)^{i+j} \det A_{ji} \right)$$

switch

$$\begin{pmatrix} 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ -11 \end{pmatrix}$$

Eigenvalues & eigenvectors

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 6 \end{pmatrix}$$

$$\begin{pmatrix} \lambda-2 & -1 & -1 \\ -2 & \lambda-3 & -2 \\ -4 & -2 & \lambda-6 \end{pmatrix}$$

$$\underbrace{\det(\lambda I - A)}_{\text{characteristic polynomial}} = (\lambda-2)(\lambda-3)(\lambda-6) - 4 - 8$$

$$-4(\lambda-3) - 2(\lambda-6) - 4(\lambda-2)$$

$$= \lambda^3 - 11\lambda^2 + 26\lambda - 16$$

$$\begin{array}{r} \lambda^2 - 10\lambda + 16 \\ \lambda - 1 \overline{) \lambda^3 - 11\lambda^2 + 26\lambda - 16} \\ -\lambda^3 + \lambda^2 \\ \hline -10\lambda^2 + 26\lambda - 16 \\ 10\lambda^2 - 10\lambda \\ \hline 16\lambda - 16 \end{array}$$

$$\lambda^2 - 10\lambda + 16 = (\lambda-2)(\lambda-8)$$

$$\boxed{\begin{array}{c} \lambda = 1 \\ \lambda = 2 \\ \lambda = 8 \end{array}}$$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 6 \end{pmatrix}$$

$$\lambda = 1 \qquad \lambda I - A = \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -4 & -2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ -4 & -2 & -5 \end{pmatrix} 4 \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \cdot 1/2$$

$$\begin{pmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1/2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}}$$

$$\lambda = 2 \quad \lambda I - A = \begin{pmatrix} 0 & -1 & -1 \\ -2 & -1 & -2 \\ 4 & -2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \end{pmatrix} \qquad \boxed{\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}}$$

$\lambda = 8$

$$\lambda E - A = \begin{pmatrix} 6 & -1 & -1 \\ -2 & 5 & -2 \\ -4 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & -5 & 2 \\ 6 & -1 & -1 \\ -4 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & -5 & 2 \\ 0 & 14 & -7 \\ \cancel{0} & \cancel{-12} & \cancel{6} \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -5/2 & 1 \\ 0 & 1 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \end{pmatrix} \qquad \boxed{\begin{matrix} 1 \\ 2 \\ 4 \end{matrix}}$$

<u>Diagonalise:</u>

$$\boxed{\begin{pmatrix} 1 & -1 & -3 \\ 2 & -2 & 1 \\ 4 & 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 & -3 \\ 2 & -2 & 1 \\ 4 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Diagonalize $A = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda+2 & -1 \\ 1 & \lambda+2 \end{pmatrix} =$$

$$= \lambda^2 + 4\lambda + 5$$

Quadratic formula

Root of $ax^2 + bx + c = 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = \underline{\underline{-2 \pm i}}$$

<u>Eigenvectors</u>: $\lambda = -2+i$ $\quad \lambda I - A = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \sim \begin{pmatrix} 1 & i \end{pmatrix}$ $\quad \boxed{\begin{pmatrix} -i \\ 1 \end{pmatrix}}$ eigenvector

real matrix $\Rightarrow$ $\lambda = -2-i$ eigenvector $\begin{pmatrix} i \\ 1 \end{pmatrix}$

Answer: $\underline{\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -2+i & 0 \\ 0 & -2-i \end{pmatrix}}$

Find the Jordan form of $A = \begin{pmatrix} 0 & 4 \\ -1 & -4 \end{pmatrix}$

$$\det \begin{pmatrix} \lambda & -4 \\ 1 & \lambda+4 \end{pmatrix} = \lambda(\lambda+4) + 4 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

$\lambda = -2$ <u>algebraic multiplicity 2</u>

<u>geometric multiplicity 1</u>

Eigenvector

$$-2I - A = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \end{pmatrix} \qquad \boxed{u = \begin{pmatrix} 2 \\ -1 \end{pmatrix}}$$

$(-2I - A)v = -u \quad \Big| \quad -2v + u = Av$

$$\left(\begin{array}{cc|c} -2 & -4 & -2 \\ 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \end{array}\right) \qquad \boxed{v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

<u>Answer:</u>

$$\boxed{\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 4 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}}$$

$J_2(-1)$

Suppose A is a matrix with two eigenvalues $\lambda = 2$, $\lambda = -3$

k	1	2	3	4	5
$\dim \ker(2I - A)^k$	4	6	7	8	8
$\dim \ker(-3I - A)^k$	3	6	7	7	

Double difference sequences:

$\lambda = 2$

$$\begin{array}{cccccc} 0 & 4 & 6 & 7 & 8 & 8 \\ & 4 & 2 & 1 & 1 & 0 \\ & & 2 & 1 & 0 & 1 \end{array}$$

# of blocks of size: 1, 2, 3, 4

$\lambda = -3$

$$\begin{array}{ccccc} 0 & 3 & 6 & 7 & 7 \\ & 3 & 3 & 1 & 0 \\ & & 0 & 2 & 1 \end{array}$$

size

Jordan form:

$$\begin{pmatrix} \boxed{2} & & & & & & & \\ & \boxed{2} & & & & & 0 & \\ & & \boxed{\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}} & & & & & \\ & & & \boxed{\begin{smallmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{smallmatrix}} & & & & \\ & & & & \boxed{\begin{smallmatrix} -3 & 1 \\ 0 & -3 \end{smallmatrix}} & & & \\ & 0 & & & & \boxed{\begin{smallmatrix} -3 & 1 \\ 0 & -3 \end{smallmatrix}} & & \\ & & & & & & \boxed{\begin{smallmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{smallmatrix}} & \end{pmatrix}$$

Answer:

(order of Jordan blocks doesn't matter)

Orthogonally diagonalise

$$A = \begin{pmatrix} 13 & 3 \\ 3 & 5 \end{pmatrix}$$

real symmetric

$$\det(\lambda I - A) = (\lambda-13)(\lambda-5)-9 = \lambda^2 - 18\lambda - 56$$
$$= (\lambda-4)(\lambda-14)$$

$\boxed{\lambda = 14}$ $\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 \end{pmatrix}$ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$

length $\sqrt{10}$

$\boxed{\lambda = 4}$ $\begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ length $\sqrt{10}$ $\begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$

Answer

B orthogonal

$$\boxed{\underbrace{\begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}}_{B^T} \begin{pmatrix} 13 & 3 \\ 3 & 5 \end{pmatrix} \underbrace{\begin{pmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}}_{B} = \begin{pmatrix} 14 & 0 \\ 0 & 4 \end{pmatrix}}$$

Orthogonally diagonalise

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

↑ real symmetric matrix

$\lambda = 1, 1, 4$

eigenvalues real

algebraic = geometric multiplicity

(no higher Jordan blocks)

$\lambda = 1 \quad (1 \;\; 1 \;\; 1)$

$$\underbrace{\begin{array}{cc} u & v \\ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{array}}_{\text{Not orthogonal!}}$$

keep v

$$u \rightsquigarrow \overbrace{u - \frac{u \cdot v}{v \cdot v} v}^{\bar{u}}$$

$$\sim (v \cdot v) u - (u \cdot v) v$$

$$2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \underset{\bar{u}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}$$

$|v| = 2 \quad |\bar{u}| = 6$

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{pmatrix}$$

$\lambda = 4$ eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Orthogonal row echelon form of

$$\begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \end{pmatrix}$$

<u>Answer</u>: OREF

$$\boxed{\begin{pmatrix} 3/\sqrt{84} & 5/\sqrt{84} & 5/\sqrt{84} & 5/\sqrt{84} \\ 0 & 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$$

$$u \longmapsto (v \cdot v)u - (u \cdot v)v$$

$$6(1\ 1\ 3\ 1) + 4(0\ 1\ -2\ 1) = (6\ 10\ 10\ 10)$$

$$\sim (3\ 5\ 5\ 5)$$

length : $\sqrt{84}$

$$\|v\| = \sqrt{6}$$