MATH 592 1/10/2024

Class policy on Canvas

Course Assistant : Ben Riley

Discussion: Tomorrow 1/11 1-2PM
Upper Atrium

A model problem:

$S^2 \not\cong S^1 \times S^1$

$n = 2$ surfaces

topological spaces

topological n-manifolds: A space X such that every point $x \in X$ has an open neighborhood $\cong \mathbb{R}^n$

Suppose we had a rubber infinitely elastic
— can shrink to a point.

If we stretch the rubber band on S^2

it can always shrink to a point while staying on S^2.

On $S^1 \times S^1$

This rubber band cannot shrink to a point

For a rigorous proof, we need rigorous definitions.

S^2 = unit sphere in $\mathbb{R}^3$

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$$

On $\mathbb{R}^{n+1}$, we have a metric. S^n has the subspace topology.

$$\cong D^n / S^{n-1}$$

$D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$

$S^{n-1} \subset D^n$

X/Y $\quad Y \subseteq X$ <u>quotient topology</u>

$U \subseteq X/Y$ open iff its inverse image in X is open

$$\begin{tikzcd} D^n/S^{n-1} \arrow[r, phantom, "\cong"] \arrow[r, bend right] & S^n \end{tikzcd}$$

$D^n \twoheadrightarrow S^n$

$S^{n-1} \longrightarrow * = (1, 0, \dots, 0)$

induced map bijective

universal property $\Rightarrow$ continuous

homeomorphism

compact $\to$ Hausdorff is homeo.

Rubber band in a space X:

A continuous map $f: S^1 \to X$.

Shrinking rubber band to a point: <u>homotopy</u>

$f, g : Z \to X$ continuous maps

A homotopy between them is a continuous map

$h: Z \times [0,1] \to X$ (write $h_t(z) = h(z,t)$), $h_0 = f$, $h_1 = g$.

The existence of a homotopy between f, g is marked by saying: f and g are <u>homotopic</u>

the homotopy

$h : f \simeq g$

$f \simeq g$ symbol

Being homotopic is an equivalence relation

"Rubber band on S^2 always shrinks to a point."

Theorem: Every continuous map $f: S^1 \to S^n$, $n>1$ is homotopic to a constant map.

Idea of proof: $S^n \setminus \{z\} \cong \mathbb{R}^n$ $Id_{\mathbb{R}^n} \simeq$ constant

$z \in S^n$ $h_t(z) = tz$

If f was not onto $\operatorname{Im} f \subseteq S^n \setminus \{a\}$

$\cong \mathbb{R}^n$

$h_t : \mathrm{Id}_{\mathbb{R}^n} \simeq$ constant $\qquad h_t \circ f$.

There do exist continuous onto maps $f : S^1 \to S^n$,

Do something to construct a homotopy $f \simeq g$, g not onto.

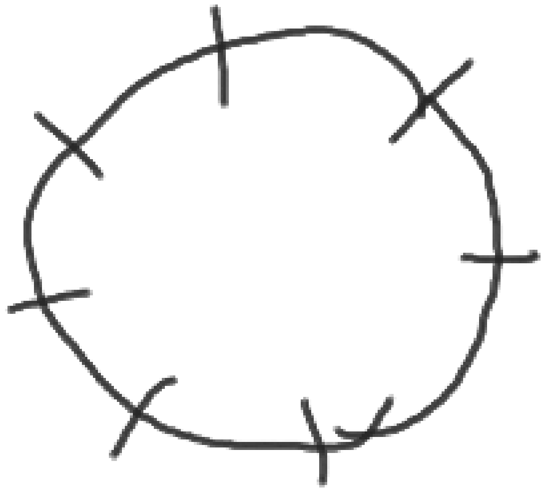
$S^n \subseteq \mathbb{R}^{n+1}$ has the induced metric.

Every continuous map where the domain between metric spaces

is compact is uniformly continuous $f: S^1 \to S^n$

$\exists N$ $\quad \|x-y\| \le \frac{2\pi}{N} \Rightarrow \|f(x)-f(y)\| < 2$

arc length metric on S^1



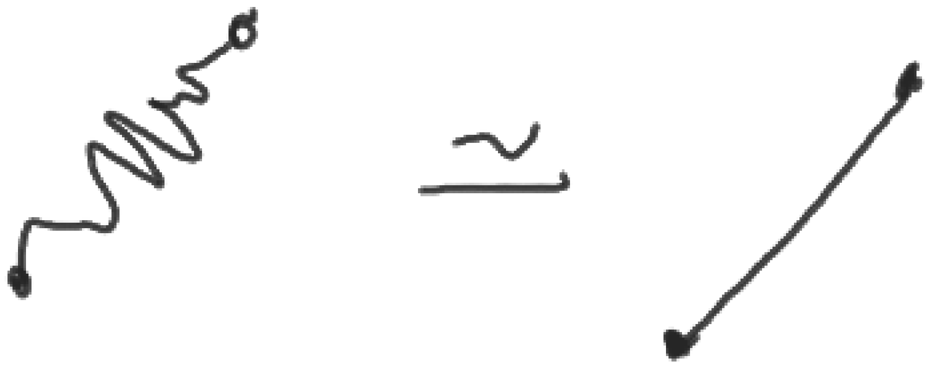
$S^1 \subseteq \mathbb{C}$

$$S^1_k = \{\exp(2\pi i t) \mid \frac{k}{N} \le t \le \frac{k+1}{N}\}$$

$F|_{S^1_k} : S^1_k \to S^n$ not onto.

$f|S^1_k \simeq$ linear map (extends to a linear map $\mathbb{R}^2 \to \mathbb{R}^{n+1}$ $\iff$ going around a great circle).

homotopy does not move end points



Can glue the homotopies to construct a homotopy $f \simeq g$ where g is linear on each S^1_k.

Claim: such a map $g : S^1 \to S^n$ cannot be onto. if $n > 1$

$S^n \neq \bigcup$ finitely many great circles

$\mathbb{R}^{n+1} \neq \bigcup$ finitely many vector subspaces of dim. 2.

$n \geqslant 2$

$n+1 \geqslant 3$

$\square$

(HW): Due 10 AM 1/17.

(late HW not accepted)

(1) Prove that a real vector space of dimension n is not a union of finitely many vector subspaces of dim. $< n$.

(2) Prove that if a continuous map $f: S^1 \to X$ is homotopic to a constant map then there exists a homotopy $h: f \simeq \text{constant}$ such that $h_t(1) = h_0(1)$

$(S^1 \subset \mathbb{C})$