MATH 592 1/12/2024

<u>Last time</u>: <u>Theorem</u>: Every continuous map $f: S^1 \to S^n$, $n > 1$ is homotopic to a constant map. $\square$

<u>Today</u>: <u>Theorem</u>: There exists a continuous map $f: S^1 \to S^1 \times S^1$ which is not homotopic to a constant map.

$\therefore S^2 \not\simeq S^1 \times S^1$

__Proof:__ $\left(S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \} \right)$

$f : S^1 \to S^1 \times S^1$ e.g. $f(x) = (x, 1)$.

$$\begin{tikzcd} S^1 \arrow[r, "{(x,1)}", "f"'] \arrow[dr, "\mathrm{Id}"'] & S^1 \times S^1 \arrow[d, "\mathrm{proj}_1"] \\ & S^1 \end{tikzcd}$$

$$(z, f) \mapsto z$$

It suffices to prove that $\mathrm{Id} : S^1 \to S^1$ is <u>not</u> homotopic to a constant map.

$$e := \exp(2\pi i t) : \mathbb{R} \xrightarrow{e} S^1 \quad \Big| \; e^{-1}(\exp(2\pi i t)) = t + \mathbb{Z}$$

For any $z \in S^1$: $e^{-1}(S^1 \setminus \{z\}) = \mathbb{R} \setminus e^{-1}(z) = \coprod_{k \in \mathbb{Z}} (t_0 + k, t_0 + k + 1)$

$t_0 \in e^{-1}(z)$

$$\coprod_{k\in\mathbb{Z}} (t_0+k,\ t_0+k+1) \xrightarrow{e|_{\dots}} S^1\setminus\{x\}$$

restricts to a homeomorphism on each of these intervals.

<u>Lemma</u>: For every continuous map $h:[0,1]\times[0,1]\to S^1$, $t_0\in e^{-1}(0,0)$, there exists a unique lift $\tilde{h}:[0,1]\times[0,1]\to\mathbb{R}$ with $\tilde{h}(0,0)=t_0$.

An analogous statement also holds with $[0,1]\times[0,1]$ replaced by $[0,1]$

Lift:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1] \times [0,1]} \arrow[ur, dashed, "\tilde{h}"] \arrow[r, "h"'] & S^1 \end{tikzcd}$$

$e \circ \tilde{h} = h$.

<u>Proof</u>: h is uniformly continuous. $\exists N \in \mathbb{N}$

$$x, y \in \underbrace{\left[\frac{k}{N}, \frac{k+1}{N}\right] \times \left[\frac{l}{N}, \frac{l+1}{N}\right]}_{I_{k,l}} \Rightarrow |h(x) - h(y)| < 2$$

$h(I_{k,\ell}) \subseteq S^1 \setminus \{z_{k,\ell}\}$.

$(e\tilde{h} = h)$

If $\tilde{h}$ is defined on some connected subset $T \subseteq I_{k,\ell}$ then it extends uniquely to $I_{k,\ell}$.

Next discussion: Th 1/8 1-2 PM

An interval is connected

A continuous image of a connected space is connected

$I_{0,0}, I_{0,1}, \ldots I_{0,N-1}, I_{1,0}, \ldots, I_{1,N-1} \ldots$

Lemma $\square$ $\ldots I_{N-1,N-1}$

$k : S^1 \times [0,1] \to S^1$

$k(z,0) = z \qquad k(z,1) = \text{constant}$

$S^1 \cong [0,1]/0\sim 1$

$j \quad j(t) = e^{2\pi i t}$

$h : [0,1] \times [0,1] \to S^1$

$h(z,t) = k(j(z),t)$

<u>lemma:</u>

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]\times[0,1]} \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & S^1 \end{tikzcd}$$

$\tilde{h}(1,0) = \tilde{h}(0,0) + 1 \;\Big|\; \tilde{h}(1,1) = \tilde{h}(0,1)$

(uniqueness of lifting on intervals

$[z,0]$, $[z,1]$

$\tilde{h} : [0,1] \times [0,1] \to \mathbb{R}$

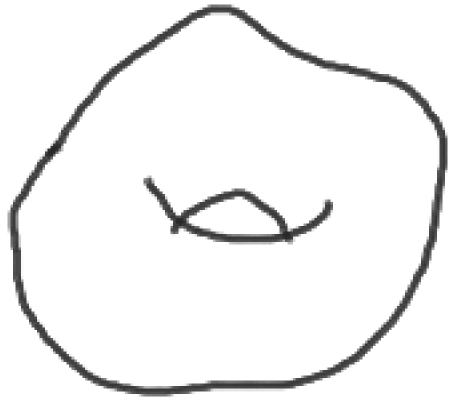
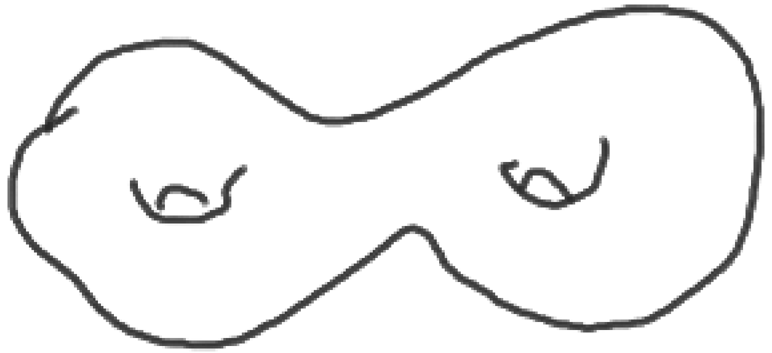
$\tilde{h}(1,0) = \tilde{h}(0,0) + 1$

$\tilde{h}(1,1) = \tilde{h}(0,1)$

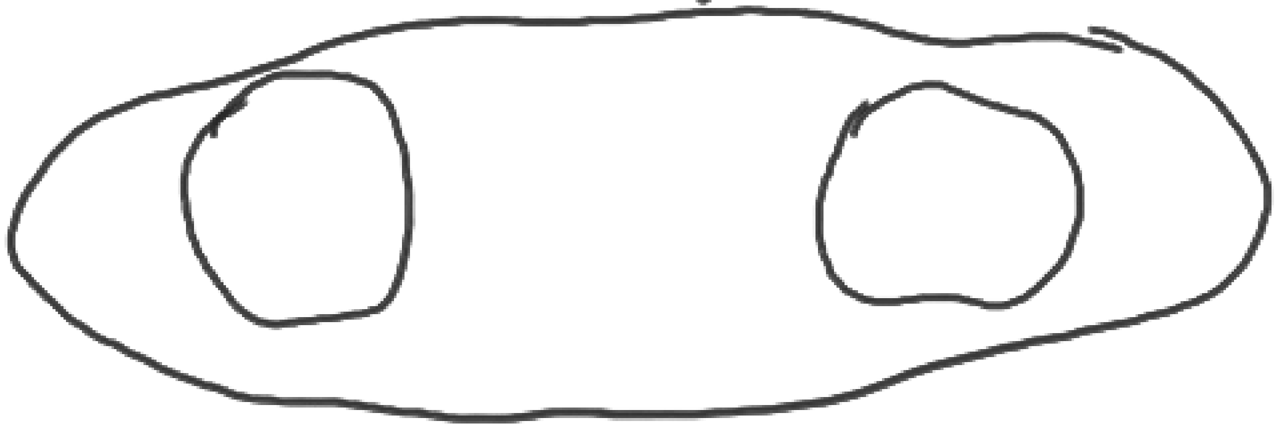
$\tilde{h}(1,t) \in \tilde{h}(0,t) + \mathbb{Z}$

$\inf \{ t \mid \tilde{h}(1,t) = \tilde{h}(0,t) \}$ $\times$ continuity $\square$

We have not done any algebra. Why algebra?

 $\not\simeq$ 

The fundamental group X



A **based space** is a space X with a chosen point $* \in X$. A **based map** $(X,*) \to (Y,*)$ is a continuous map $f : X \to Y$ such that $f(*) = *$

$f(*_X) = *_Y$.

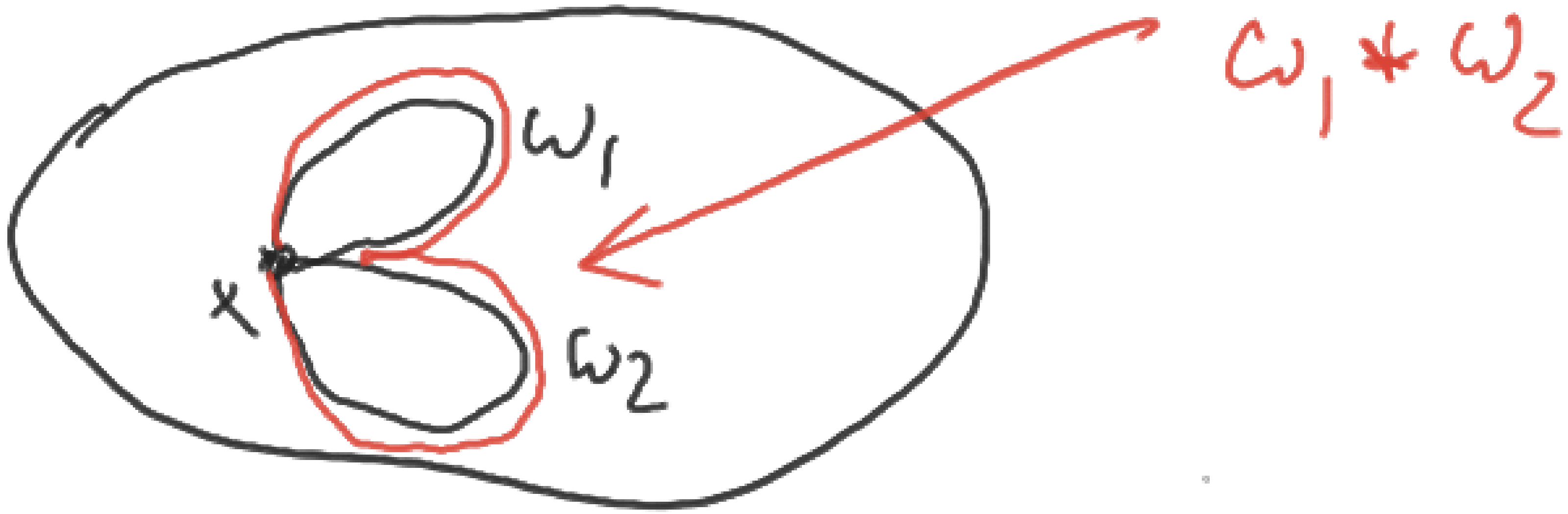
$\pi_1(X, x)$ is the set of equivalence classes
$x \in X$

of continuous based maps $\omega : (S^1, 1) \to (X, x)$

with respect to the equivalence relation of

based homotopy:

$h_t(*) = *$ $\qquad$ $h_t(1) = x$



Choose $s \in (0, 1)$

linear increasing $\cong$ $\ell_1 : [0, s] \to [0, 1]$

$\ell_2 : [s, 1] \to [0, 1]$

$$\omega_1 * \omega_2 (x) = \begin{cases} \omega_1(\ell_1(x)) & 0 \leq x \leq s \\ \omega_2(\ell_2(x)) & s \leq x \leq 1 \end{cases}$$

Associativity is obvious

$\omega_1 * \omega_2 * \omega_3$ $\qquad$ $0 < s_1 < s_2 < 1$

unit: $\eta : (S^1, 1) \to (X, x)$

$$\eta(z) = x$$

inverse: $\bar{\omega}(x) = \omega(1-x)$

(HW) (3) Prove rigorously that the definition $\omega_1 * \omega_2$ does not depend on the choice of $s \in (0,1)$ up to based homotopy.

on gradescope

(4) Prove that a product of two connected spaces is connected

Due: 1/17 10AM