MATH 592 1/17/2024

Defined fundamental group $\pi_1(X,x)$, $x \in X$ last time.

We say that a space X is <u>contractible</u> if Id_X is homotopic to a constant map.

<u>Theorem</u>: If X is contractible then $\pi_1(X,x)=0$ for any $x \in X$.

Proof: Any map $\alpha: S^1 \to X$, $(1 \mapsto x)$ is homotopic to a constant map by composing with the homotopy $Id_X \simeq$ constant. By HW2, α is homotopic to a constant map via a homotopy h: $h_t(1)=x$. $[\alpha]=e \in \pi_1(X,x)$. $\square$

the equivalence class corresponding to α.

Next Discussion: Thursday 1/18 1-2PM Upper Atrium
(moving following discussion to Tuesdays 1-2PM).
Next discussion topic: category theory.

<u>Theorem</u>: $\pi_1(S^1, 1) \cong \mathbb{Z}$

base point does not matter.

Proof: (using the model of S^1 as unit circle in $\mathbb{C}$)

$z^n : S^1 \longrightarrow S^1$
$z \longmapsto z^n$

is a homomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$

$$z^m *_1 z^n \stackrel{?}{=} z^{m+n} \qquad m, n \in \mathbb{N}$$

$\bar{z} = z^{-1}$ inverse to z

Injective: Recall from last time

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] & S^1 \end{tikzcd} \qquad \begin{matrix} t \\ \downarrow \\ e^{2\pi i t} \end{matrix}$$

$[0,1] \times [0,1]$

lifts to $\tilde{g} : [0,1] \to \mathbb{R}$ $\quad e\tilde{g} = g$

resp. $[0,1]\times[0,1]$

$\tilde{g}$ unique subject to choosing

$$\tilde{g}(0) \in e^{-1}(g(0))$$

$\qquad (0,0) \qquad\qquad (0,0)$

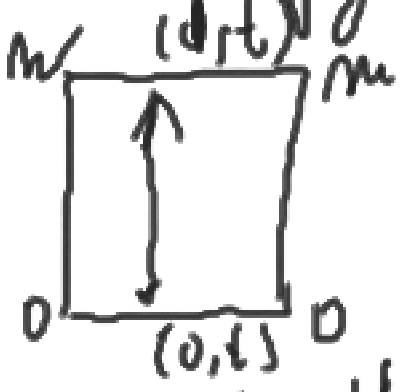
$z^n \quad \tilde{z}^n = n : [0,1] \to \mathbb{R}$

$\qquad t \longmapsto nt$

If $h : z^n \simeq z^m$ $\quad m \neq n$ then lift the homotopy: $\tilde{h}(0,0) = 0$

$\qquad s\ t$

homotopy coordinate second



$\tilde{h}(1,t) - \tilde{h}(0,t) \in \mathbb{Z}$

Take $\sup\{t \mid \tilde{h}(1,t) - \tilde{h}(0,t) = n\}$. Not continuous at this point.

Showing that $\mathbb{Z} \longrightarrow \pi_1(S^1, 1)$

$n \longmapsto z^n$

<u>is onto</u>: Suppose $\alpha : [0,1]/_{0\sim 1} \longrightarrow S^1$

$\alpha(0) = 1$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[r, "\alpha"'] \arrow[ur, dotted, "\tilde\alpha"] & S^1 \end{tikzcd}$$

$\tilde\alpha(0) = 0$. Let $n = \alpha(1)$. Form $z^{-n} * \alpha = \beta$.

Do the same thing:

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1]} \arrow[r, "\beta"'] \arrow[ur, dashed, "\tilde\beta"] & S^1 \end{tikzcd}$$

$\tilde\beta(0) = 0$. Then also $\tilde\beta(1) = 0$. | Can write $\tilde\beta : [0,1]/_{0\sim 1} \longrightarrow \mathbb{R}$

$\pi_1(\mathbb{R}, 0) = 0$ $\tilde h\ \tilde\beta \simeq \mathrm{const}_0$ $\tilde h_t(0) = 0$ Set $h = e \circ \tilde h$

h is a homotopy between $z^{-n} * \alpha$ and constant 1.

$[0,1]/_{0\sim 1} \longrightarrow S^1$

$h_t(0) = 1$ by definition.

$$\therefore [z^n] * \alpha = 0 \in \pi_1(S^1, 1)$$

$$\therefore \alpha = -n \in \pi_1(S^1, 1) = \mathbb{Z}. \quad \square$$

The language of category theory

models the general properties of sets and maps

A category $\mathcal{C}$ consists of two classes $\mathrm{Ob}(\mathcal{C})$, $\mathrm{Mor}(\mathcal{C})$

objects, morphisms

$S, T : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Obj}\,\mathcal{C}$

source target

$\mathrm{Id} : \mathrm{Obj}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}$

$X \longmapsto \mathrm{Id}_X$

$\left.\begin{array}{l} S\,\mathrm{Id}_X = X \\ T\,\mathrm{Id}_X = X \end{array}\right\}$ axioms

If $f \in \mathrm{Mor}\,\mathcal{C}$ $Sf = X, Tf = Y$ write $f : X \to Y$

Composition : $f : X \to Y, \; g : Y \to Z \longmapsto g \circ f : X \to Z$

<u>Axioms</u>: Associative, unital on both sides. $\mathrm{Mor}_{\mathcal{C}}(X, Y) = \{f : X \to Y\}$ is a set.

<u>Examples</u>: Top

$\mathrm{Obj}(\mathrm{Top})$ = Topological spaces

$\mathrm{Mor}(\mathrm{Top})$ = continuous maps

Based

Obj = Based spaces $(X, *)$

Mor = continuous maps $* \longmapsto *$

Grp Obj = groups Mor = homomorphisms

Ab Obj = Ab. groups Mor = homomorphisms

We call a category $\mathscr{C}$ small if $Obj(\mathscr{C})$, $Mor(\mathscr{C})$ are sets.

<u>Functor</u> $F : \mathscr{C} \to \mathscr{D}$ where $\mathscr{C}, \mathscr{D}$ are categories

maps $F : \begin{cases} Obj\, \mathscr{C} \to Obj\, \mathscr{D} \\ Mor\, \mathscr{C} \to Mor\, \mathscr{D} \end{cases}$ preserve S, T, Id, composition

$$SF(f) = F(S(f)) \qquad TF(f) = F(T(f)) \qquad F(Id_X) = Id_{F(X)}$$

When $g \circ f$ is defined, $F(g) \circ F(f) = F(g \circ f)$.

HW

1. Calculate $\pi_1(X \times Y, (x,y))$ in terms of $\pi_1(X,x)$ and $\pi_1(Y,y)$.

2. Show how a group G becomes a category $\mathcal{G}$ with $\mathrm{Obj}\,\mathcal{G} = \{*\}$, $\mathrm{Mor}\,\mathcal{G} = G$ using the composition in G. For $\mathcal{H}$ corresponding to a group H, what is a functor $F: \mathcal{G} \to \mathcal{H}$ in terms of group theory?

Due Monday 1/22 10 AM