MATH 592 1/19/2024

Based = Spaces with a base-point $*$, continuous maps

$f: X \to Y \quad f(*) = *$.

Grp = Groups, homomorphisms of groups

Next discussion: Tuesday, EH 5822
1-2 PM

$$\underbrace{\pi_1 : \text{Based} \longrightarrow \text{Grp}}_{\text{functor}}$$

hTop = topological spaces, homotopy classes of (continuous) maps

Notation:

$$X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$$

$$f_1 \simeq f_2 \ , \ g_1 \simeq g_2 \quad \Rightarrow \quad g_1 \circ f_1 \simeq g_2 \circ f_2$$

$g = f^{-1}$

Isomorphism in a category is a morphism $f: X \to Y$ such that there exists an inverse $g: Y \to X$: $f \circ g = Id_Y$, $g \circ f = Id_X$.

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homotopy category of spaces

Isomorphism in hTop is called a homotopy equivalence

$f: X \to Y$ $\quad g: Y \to X$ continuous maps $\quad f \circ g \simeq Id_Y$, $g \circ f \simeq Id_X$

A space homotopy-equivalent to a point is called contractible.

Groupoid = A category where every morphism is an isomorphism.

Example : Let X be a topological space.

$\pi(X)$ Objects = points of X (X as a set)

groupoid

Morphisms $\omega : x \to y$

maps $\omega : [0,1] \to X$

$0 \longmapsto x$

$1 \longmapsto y$

/ $\omega_1 \sim \omega_2$ when there exists a homotopy $h : \omega_1 \simeq \omega_2$

$h_t(0) = x$

$h_t(1) = y$ $\quad \forall t \in [0,1]$

Composition is defined the same as in the fundamental group

$\omega_1 : x \to y \qquad \omega_2 : y \to z \qquad \omega_1 * \omega_2$ (as in the def. of $\pi_1(X,x)$)

Natural transformation: Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors.

A natural transformation $\varphi : F \to G$ consists of a $\mathcal{D}$-morphism, for every object $x \in \mathcal{C}$

$$\varphi_x : F(x) \to G(x)$$

such that for a $\mathcal{C}$-morphism $f : x \to y$ we have a commutative diagram

$$\begin{tikzcd} F(x) \arrow[r, "\varphi_x"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\varphi_y"] & G(y) \end{tikzcd}$$

means:

$$G(f) \circ \varphi_x = \varphi_y \circ F(f)$$

Example with the fundamental groupoid:

$\pi : \mathrm{Top} \longrightarrow \mathrm{Groupoids}$ ⟵ (small) groupoids, functors

(on morphisms $f : X \to Y$

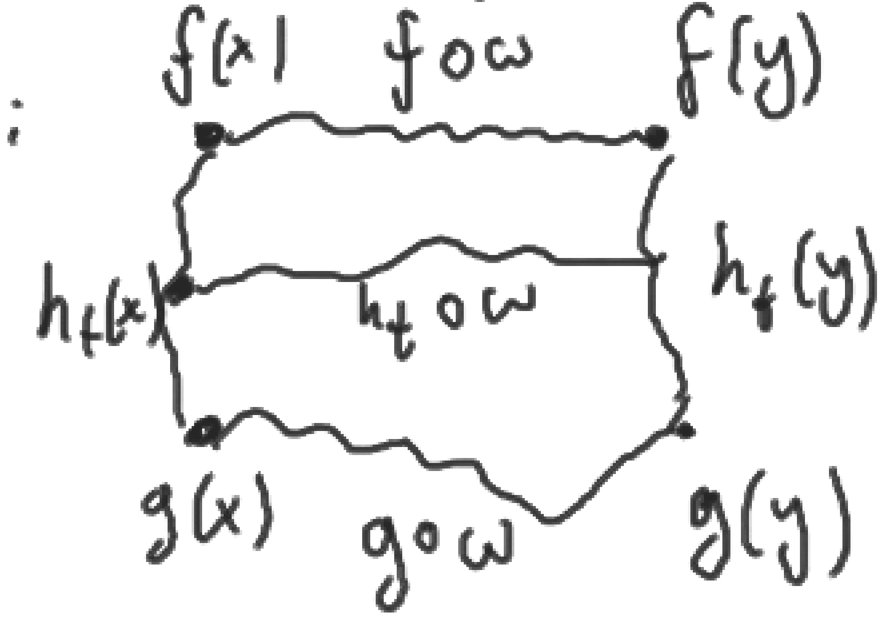
$\omega : x_1 \longrightarrow x_2 \longmapsto f \circ \omega$).

$\omega : [0,1] \to X$

Proposition: A homotopy $h : f \simeq g$, $f, g : X \to Y$ specifies a natural transformation $\pi(f) \longrightarrow \pi(g)$.

$\underbrace{\pi(f) : \pi(X) \to \pi(Y) \quad \pi(g)}_{\text{functors}}$

Proof:



$\omega : x \to y \in \pi(X)$

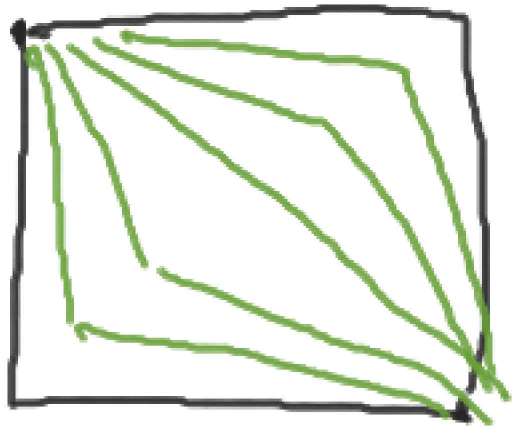
$[0,1] \longrightarrow X$

$\omega(0) = x \quad \omega(1) = y$

$[0,1] \times [0,1] \longrightarrow Y$

$(s,t) \longmapsto h_t\,\omega(s)$

Define $\varphi : \pi(f) \to \pi(g)$ by: $\varphi_x = h_t(x)$



We say that a natural transformation φ is a natural isomorphism if φ_X is an isomorphism for every X. (Note: $\varphi^{-1} = (\varphi_X^{-1})$ is automatically natural). Mac Lane: Categories for a working mathematician.

In a groupoid, every natural transformation is a natural isomorphism.

We can form a category Grp° whose objects are small groupoids and morphisms are equivalence classes of functors under the relation of being naturally isomorphic.

$$\begin{tikzcd} A \arrow[r, bend left=60, "F"{name=U}] \arrow[r, bend right=60, "G"'{name=D}] \arrow[from=U, to=D, phantom, "\varphi\Downarrow"] & B \arrow[r, bend left=60, "F'"{name=U2}] \arrow[r, bend right=60, "G'"'{name=D2}] \arrow[from=U2, to=D2, phantom, "\psi\Downarrow"] & C \end{tikzcd}$$

$$F' \circ F \xrightarrow{\psi \circ \varphi} G' \circ G$$

$$\begin{tikzcd} A \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\varphi\Downarrow"] & B \arrow[r, bend left=50, "F'"] \arrow[r, bend right=50, "G'"'] \arrow[r, phantom, "\psi\Downarrow"] & C \end{tikzcd}$$

$x \in A$ $\qquad F(x) \xrightarrow{\varphi_x} G(x)$

$$\begin{tikzcd} & G'F(x) \arrow[dr, "G'(\varphi(x))"] & \\ F'F(x) \arrow[ur, "\psi_{F(x)}"] \arrow[rr] \arrow[dr, "F'(\varphi_x)"'] & & G'G(x) \\ & F'G(x) \arrow[ur, "\psi_{G(x)}"'] & \end{tikzcd}$$

An equivalence of categories $\mathcal{C}, \mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong Id_{\mathcal{C}}$

$FG \cong Id_{\mathcal{D}}$

naturally isomorphic

The <u>proposition implies</u>: We have a functor

$$\Pi : hTop \longrightarrow Groupoid^{\circ}$$

$X \simeq Y$

<u>Corollary</u>: If X and Y are homotopy equivalent spaces then $\Pi(X)$ and $\Pi(Y)$ are equivalent categories

HW: 3 Will be on gradescope

on introducing $\pi_n(X, x)$ $n > 1$.

Theorem: $\pi_1(S^n, *) = 0$ $n > 1$

any base point

Proof.
We have proved $S^1 \to S^n \simeq$ constant map.
By the HW, the statement follows. $\square$

Why is S^n not contractible?
We will need to use π_n.