MATH 592 1/22/2024

<u>Equivalence of categories</u>: $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$,

$GF \cong \mathrm{Id}_{\mathcal{C}}$, $FG \cong \mathrm{Id}_{\mathcal{D}}$

natural isomorphism

Discussion: Tomorrow 1/23 1-2 PM EH 5822

Cat^{o} Objects = small categories

Morphisms = functors up to natural isomorphism

Isomorphism in Cat^{o} = equivalence of categories

$\mathrm{Groupoid}^{o}$ Objects = groupoids $\mathrm{Mor}_{\mathrm{Groupoid}^{o}}(x,y) = \mathrm{Mor}_{\mathrm{Cat}^{o}}(x,y)$ x, y groupoids ← full subcategory

We proved that

$$\Pi : \mathrm{hTop} \longrightarrow \mathrm{Groupoid}^{\circ}$$

fundamental groupoid

spaces, homotopy classes of maps

$\therefore$ Homotopy-equivalent spaces have equivalent fundamental groupoids.

More on equivalence of categories:

Theorem: Let $\mathcal{C}$ be any category, and let $\mathcal{C}_0$ be a full subcategory on a class of objects of $\mathcal{C}$ which contains precisely one representative from each isomorphism class of objects. Then $\mathcal{C}$ and $\mathcal{C}_0$ are equivalent.

$\mathcal{C}_0$ is called a skeleton of $\mathcal{C}$.

Proof sketch: $\mathscr{C}_0 \xrightarrow{F} \mathscr{C}$

? $G : \mathscr{C} \to \mathscr{C}_0$. For each object x, choose an isomorphism $\gamma_x : x \xrightarrow{\cong} x_0$ where $x_0 \in \mathrm{Obj}\,\mathscr{C}_0$. (Technical advice: choose $\gamma_x = \mathrm{Id}$ if $x \in \mathrm{Obj}\,\mathscr{C}_0$.)

$G(x) = x_0$ $\qquad f : x \to y \qquad$ $G(f)$:

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[d, "\gamma_y"', "\cong"] \\ x_0 \arrow[u, "\gamma_x^{-1}", "\cong"'] & y_0 \end{tikzcd}$$

<u>HW</u> 1: Finish the proof that this pair F, G is an equivalence of categories.

What is the skeleton of a groupoid Γ? Choose one object in each isomorphism class: $\mathrm{Obj}\,\Gamma_0$.

a group is a groupoid with a single object

$$\mathrm{Mor}\,\Gamma_0 = \coprod_{x \in \mathrm{Obj}\,\Gamma_0} \mathrm{Aut}_\Gamma(x)$$

the group of all $\overbrace{\text{isomorphisms}}^{\text{iso-}}$ $x \to x$ (but we are in a groupoid)

disjoint union of categories: no morphisms between the different summands.

Applying to the fundamental groupoid $\pi(X)$:

$$\mathrm{Obj}\,\pi(X)_0 = \pi_0 X = \{\text{path-connected components of } X\}$$

just a set

$$\| \\ X/x \sim y \text{ if there is a path from } x \text{ to } y.$$

To recap: For a space X, choose a set $X_0 \subseteq X$ where X_0 contains exactly one point in each path-component of X. $(X_0 \cong \pi_0 X)$

Then we can write

$$\Pi(X) = \coprod_{x \in X_0} \pi_1(X, x).$$

A space X is called path-connected if it has only one path-component.

$(\pi_0 X \cong \{*\})$

We see that if X is path-connected, then πX is equivalent to $\pi_1(X,x)$ for any $x \in X$.

What is equivalence of groups? From HW, a functor on groups is just a homomorphism of groups.

<u>HW</u> 2 : Prove that for two homomorphisms $f, g : G \to H$ a natural transformation $f \to g$ (thought of as functors) is the same thing as specifying an element $\alpha \in H$ such that

$$(\forall x \in G) \quad g(x) = \alpha^{-1} f(x) \alpha$$

<u>Conjugate homomorphisms.</u>

Observation: a conjugate of an isomorphism is an isomorphism.

<u>Theorem</u>: π_1 gives a functor from the full subcategory of hTop on path-connected spaces to Grp^{o}. $\square$

Objects: groups
Morphisms: conjugacy classes of homomorphisms.

If X is path-connected, I can say: Calculate $\pi_1(X)$.

determined up to $\cong$,
$\cong$ for different base points determined up to conjugation; depends on the choice of path between them.

<u>The real question</u>: Calculate $\pi_1(X)$ for more general spaces than $X = S^n$: $\pi_1(S^1) = \mathbb{Z}$

$\pi_1(S^n) = 0 \quad n>1$.

?

$\pi_1\Big($  $\Big)$, $\pi_1\Big($  $\Big)$.

The fundamental groupoid also helps with this.

Language of categories enables the discussion of <u>universal properties</u>.

<u>Example</u> (immediately needed, general discussion later): <u>Pushout</u>.

Suppose we have a category $\mathcal{C}$. Given a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "g"'] & X_2 \\ X_3 & \end{tikzcd}$$

We are trying to define a "universal" commuting square

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & X_2 \arrow[d, dashed] \\ X_3 \arrow[r, dashed] & Y \end{tikzcd}$$

← pushout diagram

We write $Y = X_3 \amalg_{X_1} X_2$

What does universality (defining the pushout) mean?

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "g"'] & X_2 \arrow[d, "\tilde{g}"] \\ X_3 \arrow[r, "\tilde{f}"'] & Y \end{tikzcd}$$

Commutes: $\tilde{g} \circ f = \tilde{f} \circ g$

Whenever we have a diagram

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "g"'] & X_2 \arrow[d, "\tilde{g}"] \arrow[dr, "g'"] & \\ X_3 \arrow[r, "\tilde{f}"] \arrow[drr, "f'"'] & Y \arrow[dr, dotted, "\exists!\, q"] & \\ & & Z \end{tikzcd}$$

$g' \circ f = f' \circ g$

there exist a unique $q : Y \to Z$ such that

$q \circ \tilde{g} = g'$, $q \circ \tilde{f} = f'$

HW (3): Pushout in sets and topological spaces

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \\ Z & & \end{array}$$

In sets, $Y \overset{f\quad g}{\sqcup_X} Z = Y \sqcup Z \big/ f(x) \sim g(x)$ for $x \in X$

In Top, same thing with the quotient topology.

When $X = U \cup V$, U, V open: Then $X = U \sqcup_{U \cap V} V$.

pushout.

Theorem 1: The fundamental groupoid in this situation preserves pushouts.

$$\pi(X) = \pi(U) \sqcup_{\pi(U \cap V)} \pi(V).$$

Theorem: In the above situation, assume $U, V, U \cap V$ are path-connected.
Let $x \in U \cap V$. Then $\pi_1(X, x)$ is the pushout of groups
$\pi_1(U, x) \amalg_{\pi_1(U \cap V, x)} \pi_1(V, x)$.

Seifert - Van Kampen Theorem.

usual notation: $*$

Follow-up question: What does the pushout of groups look like?

This theorem "completely solves" the question of computing π_1.

Types of spaces for which we can ask the question: CW-complexes.